Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Structure preserving model reduction of large-scale second-order linear dynamical systems

Tsuyoshi Koyama, Sanjay Govindjee

Center for Mechanics, Institute for Mechanical Systems

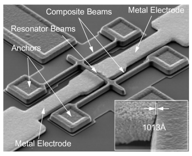
Model reduction

•Second order linear dynamical systems

They appear in many physical systems of interest such as electrical, mechanical, electromechanically coupled, and thermomechanically coupled systems (MEMS).

Single input single output system

$$\mathbf{M}\ddot{\mathbf{z}}(t)+\mathbf{D}\dot{\mathbf{z}}(t)+\mathbf{K}\mathbf{z}(t) = \mathbf{b}u(t)$$
$$y(t) = \mathbf{l}^*\mathbf{z}(t)$$



SEM of 10.47 MHz lateral free-free beam resonator
(Ref: W.-T. Hsu et al, Transducer 2001)

•Transfer function

$$H(s) = \mathbf{l}^*\left(s^2\mathbf{M}+s\mathbf{D}+\mathbf{K}\right)^{-1}\mathbf{b}$$

Matrices involved are large. Computationally expensive to evaluate.

Select small projection subspace X and Y which accurately represents the motion in a certain frequency range.

$$\mathbf{M}_R=\mathbf{Y}^*\mathbf{M}\mathbf{X} \quad \mathbf{b}_R=\mathbf{Y}^*\mathbf{b}$$
$$\mathbf{D}_R=\mathbf{Y}^*\mathbf{D}\mathbf{X} \quad \mathbf{l}_R=\mathbf{X}^*\mathbf{l}$$
$$\mathbf{K}_R=\mathbf{Y}^*\mathbf{K}\mathbf{X}$$

$$H_R(s) = \mathbf{l}_R^*\left(s^2\mathbf{M}_R+s\mathbf{D}_R+\mathbf{K}_R\right)^{-1}\mathbf{b}_R$$

Matrices involved are small!! Computationally cheap to evaluate!!

Proper selection of $\mathbf{X}$ and $\mathbf{Y}$ is crucial to preserve properties of the transfer function such as structure and approximation accuracy.

Selection of projection subspace

•Second-order Krylov subspaces for X and Y

Given matrices $\mathbf{A}$, $\mathbf{B}$ and starting vector $\mathbf{r}$, the kth second-order Krylov subspace $\mathcal{G}_k(\mathbf{A},\mathbf{B},\mathbf{r})$ is defined by the recursion relation,

$$\mathbf{r}_0 = \mathbf{r}$$
$$\mathbf{r}_1 = \mathbf{A}\mathbf{r}_0$$
$$\mathbf{r}_i = \mathbf{A}\mathbf{r}_{i-1}+\mathbf{B}\mathbf{r}_{i-2} \quad (2\le i\le k-1)$$

k iterations of the SOAR iterations

Select $\mathbf{X}$ to span the right second-order Krylov subspace

$$\mathcal{G}_k\left(\mathbf{K}^{-1}\mathbf{D},\mathbf{K}^{-1}\mathbf{M},\mathbf{K}^{-1}\mathbf{b}\right)$$

Select $\mathbf{Y}$ to span the left second-order Krylov subspace

$$\mathcal{G}_k\left(\mathbf{K}^{*,-1}\mathbf{D}^*,\mathbf{K}^{*,-1}\mathbf{M},\mathbf{K}^{*,-1}\mathbf{l}\right)$$

All computational effort of the reduced order modeling is spent on generating these subspaces, i.e., conducting SOAR iterations.

Moment matching theorem

•Moments of the transfer function

Given a transfer function, the moments are defined as the coefficients of its power series expansion around a given point.

$$H(s) = \sum_{i=0}^{\infty} M_i s^i$$
$$H_R(s) = \sum_{i=0}^{\infty} M_{Ri} s^i$$

Power series expansion around s=0. M_i and M_{Ri} are the moments.

•Theorem

Let integers $k, r \ge 0$. If,

$$\mathcal{G}_k\left(\mathbf{K}^{-1}\mathbf{D},\mathbf{K}^{-1}\mathbf{M};\mathbf{K}^{-1}\mathbf{b}\right)\subset \text{span}(\mathbf{X})$$
$$\mathcal{G}_r\left(\mathbf{K}^{*,-1}\mathbf{D}^*,\mathbf{K}^{*,-1}\mathbf{M}^*;\mathbf{K}^{*,-1}\mathbf{l}\right)\subset \text{span}(\mathbf{Y})$$

X contains the kth right second-order Krylov subspace.
Y contains the rth left second-order Krylov subspace.

then,

$$M_i = M_{Ri} \quad (0\le i\le k+r-1)$$

The first k+r moments of the transfer functions match

and,

$$H(s) = H_R(s)+\mathcal{O}\left(s^{k+r}\right)$$

•Corollary

The theorem above holds for a nonzero expansion point $s=s_0$ with minor modification of the construction of the second-order Krylov subspaces.

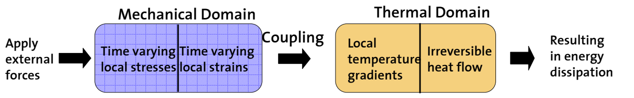
The moment matching property is a measure of the accuracy obtained from this Krylov subspace based reduced order modeling scheme. More matched moments lead to a more accurate representation of the dynamical system.

Linearized thermoelasticity

•Modeling of thermoelastic damping

Linearized thermoelasticity, which is the equations of motion coupled with the heat equation, can be used to model an energy dissipation mechanism in MEMS called Thermoelastic damping .

Energy loss from coupling of domains



•Structure of equations

$$\begin{bmatrix}\mathbf{M}_{uu} & 0\\ 0 & 0\end{bmatrix}\begin{pmatrix}\ddot{\mathbf{u}}\\ \ddot{\theta}\end{pmatrix}+\begin{bmatrix}0 & 0\\ \mathbf{D}_{tu} & \mathbf{D}_{tt}\end{bmatrix}\begin{pmatrix}\dot{\mathbf{u}}\\ \dot{\theta}\end{pmatrix}+\begin{bmatrix}\mathbf{K}_{uu} & \mathbf{K}_{ut}\\ 0 & \mathbf{K}_{tt}\end{bmatrix}\begin{pmatrix}\mathbf{u}\\ \theta\end{pmatrix} = \begin{pmatrix}\mathbf{b}_u\\ \mathbf{b}_t\end{pmatrix}u(t)$$

- **Mass and damping matrices are singular and unsymmetric.**
- **Damping and stiffness matrices are unsymmetric.**
- **The coupling terms are tranposes of each other, $\mathbf{K}_{ut}=-\mathbf{D}_{tu}^T$.**

Structure preserving model reduction

•Assume symmetric purely mechanical excitation

In practical applications, the devices of interest are excited purely mechanically in the form of electrostatic forces in the case of MEMS. Additionally, the displacement response at the loading point is usually the output quantity of interest.

$$\mathbf{l}=\mathbf{b}=\begin{bmatrix}\mathbf{b}_u\\ \mathbf{0}\end{bmatrix}$$

➡ Right and left second-order Krylov subspaces are related.

$$\mathcal{G}_k\left(\mathbf{K}^{-1}\mathbf{D},\mathbf{K}^{-1}\mathbf{M};\mathbf{K}^{-1}\mathbf{b}\right)=\text{span}\left(\left[\begin{matrix}\mathbf{r}_i^u\\ \mathbf{r}_i^t\end{matrix}\right]_{0<i<k-1}\right) \qquad \mathcal{G}_k\left(\mathbf{K}^{*,-1}\mathbf{D}^*,\mathbf{K}^{*,-1}\mathbf{M}^*;\mathbf{K}^{*,-1}\mathbf{l}\right)=\text{span}\left(\left[\begin{matrix}\mathbf{r}_i^u\\ -\sum_{j=0}^{i}\frac{1}{s_0^{j+1}}\mathbf{r}_{i-j}^t\end{matrix}\right]_{0\le i\le k-1}\right)$$

By selecting $\mathbf{X}$ and $\mathbf{Y}$ as,

$$\mathbf{X} = \begin{bmatrix}\text{span}\left\{\mathbf{r}_0^u,\ldots,\mathbf{r}_{k-1}^u\right\} & 0\\ 0 & \text{span}\left\{\mathbf{r}_0^t,\ldots,\mathbf{r}_{k-1}^t\right\}\end{bmatrix}$$
$$\mathbf{Y} = \overline{\mathbf{X}}$$

$\mathbf{X}$ and $\mathbf{Y}$ contain the right and left second-order Krylov subspaces.

- **The structure of the equation is preserved.**
- **By use of the theorem presented, one can match at least 2k moments by just producing a kth second-order Krylov subspace. This cannot be proven with the version of the theorem presented for the first-order form of the transfer function.**

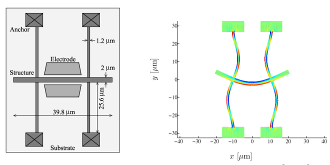
Higher accuracy!

(Ref: R.-C. Li and Z. Bai, Structure-preserving model reduction using Krylov subspace formulation, 2005)

Numerical examples

• Michigan Free-Free beam structure. (Ref: W.-T. Hsu et al, Transducer 2001)

- Material is Polysilicon
- 2D plane stress analysis
- Expand ROM at $s_0 = i9.5926\times10^6$

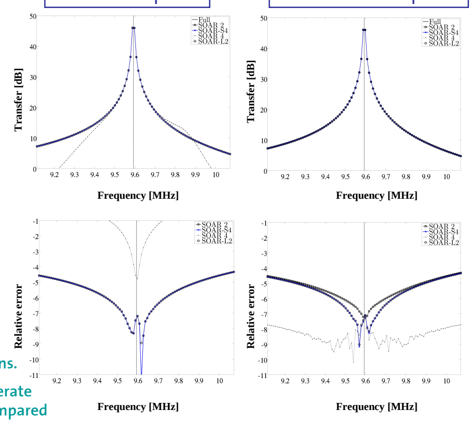


Schematic of structure and deformed shape for mode at 100 [MHz]
(Color represents temperature fluctuation).

Transfer function

- Full model: 60318 DOFs
- ROMs
  - SOAR 2: 2DOFs
    2 SOAR iterations
  - **SOAR-S4: 4DOFs**
    2 SOAR iterations
    Structure preserving model reduction
  - SOAR 4: 4DOFs
    4 SOAR iterations
  - SOAR-L2: 2DOFs
    2 SOAR iterations for both left and right second-order Krylov subspace for total of 4 SOAR iteration.

Dimensional equation | Non-dimensional equation



- **SOAR-S4 is accurate irrespective of dimensionalization**
- **SOAR-S4 requires less SOAR iterations.**
- **SOAR-S4 requires 12 seconds to generate the transfer function (100 points) compared to 165 seconds for the full model.**

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