Modeling of Thermoelastic Damping in MEMS Resonators

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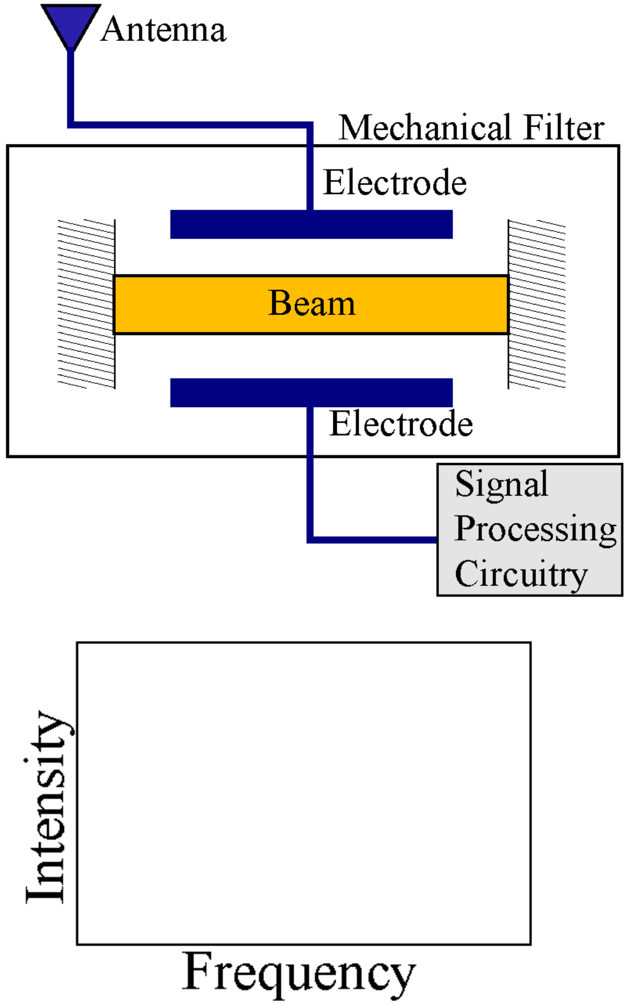
MEMS Resonators

- Micron size devices
  - Mechanical Filters in IC



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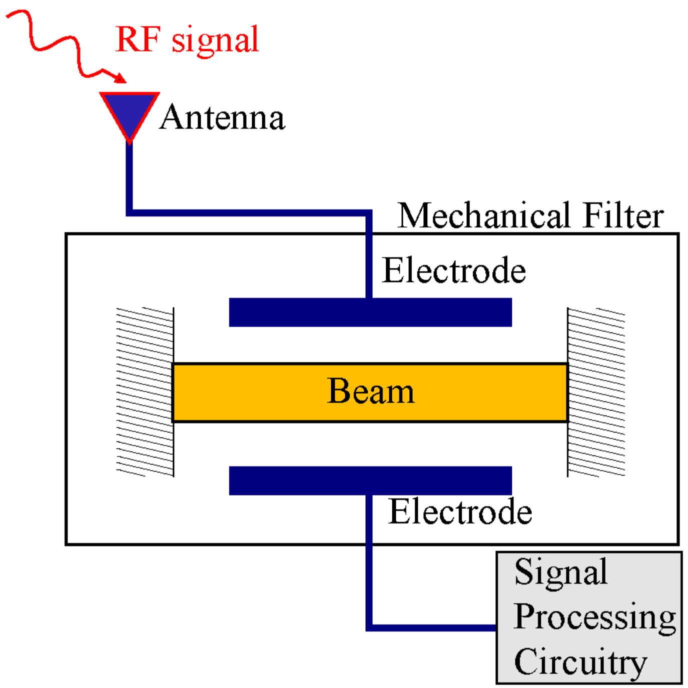

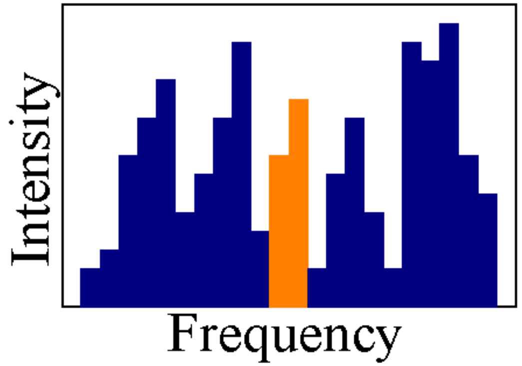

MEMS Resonators

- Micron size devices.
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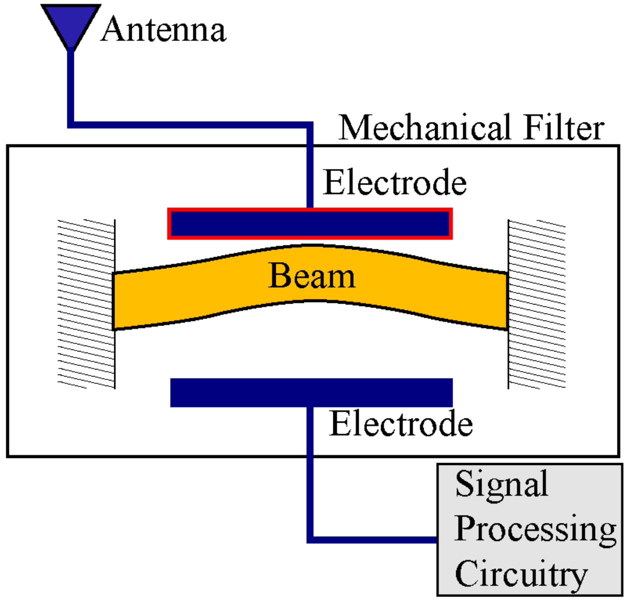
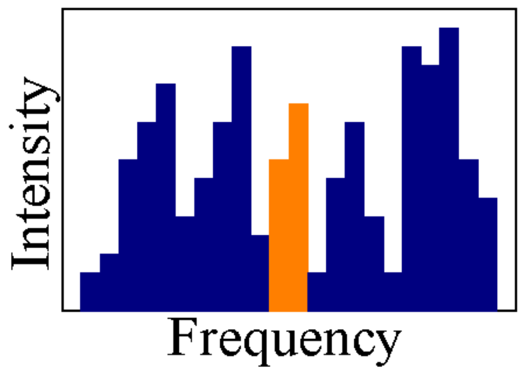
  - Electrostatically actuated





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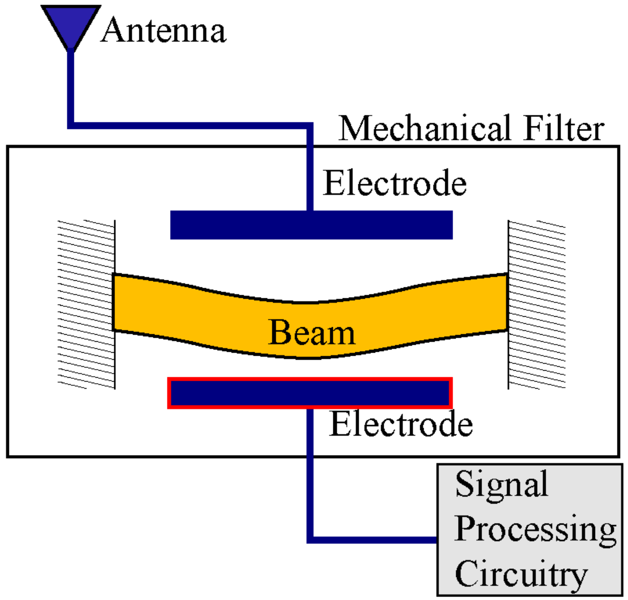
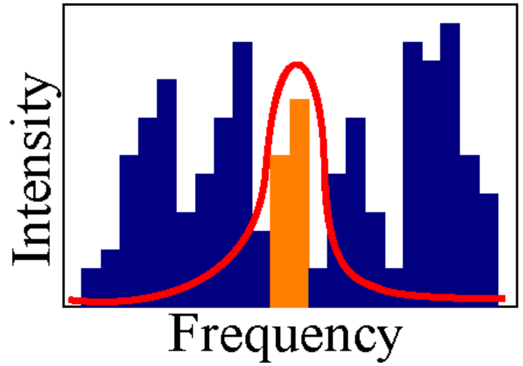
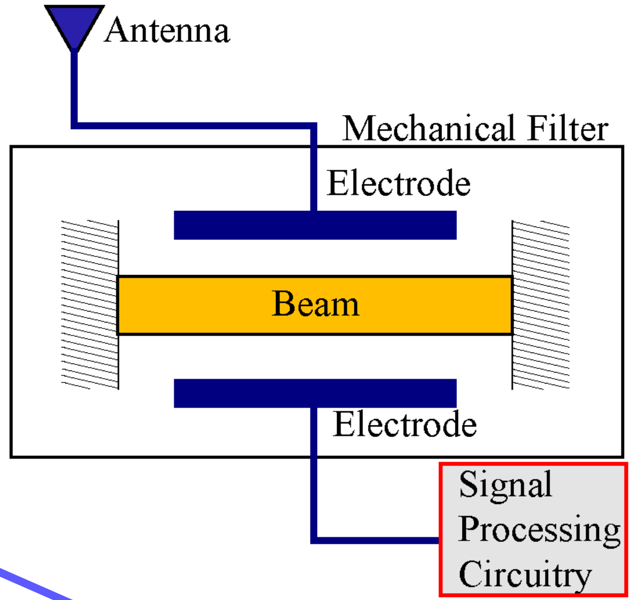


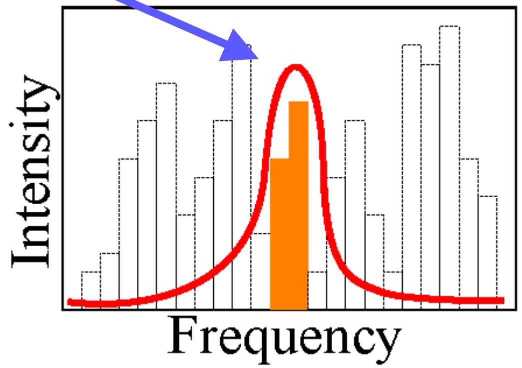


MEMS Resonators

- Micron size devices.
  - Mechanical Filters in IC
  - Electrostatically actuated
- The Quality factor (*Q*)

  Index of how narrow and sharp the peak is.





MEMS Resonators

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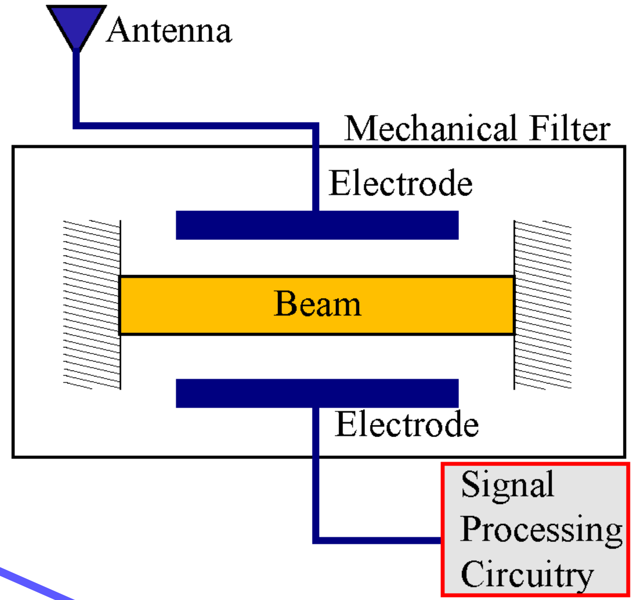
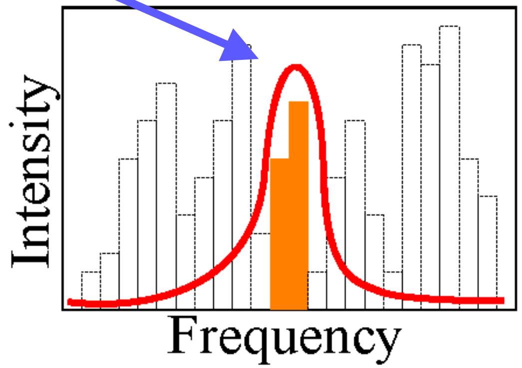


$$
\begin{aligned}
Q^{-1} &= \frac{\Delta E_{\text{dissipated energy/rad}}}{E_{\text{max stored energy}}} \\
&\approx Q^{-1}_{\text{anchor loss}} + Q^{-1}_{\text{ted}} + Q^{-1}_{\text{air}} \\
&\quad + Q^{-1}_{\text{ohmic}} + Q^{-1}_{\text{other}}
\end{aligned}
$$



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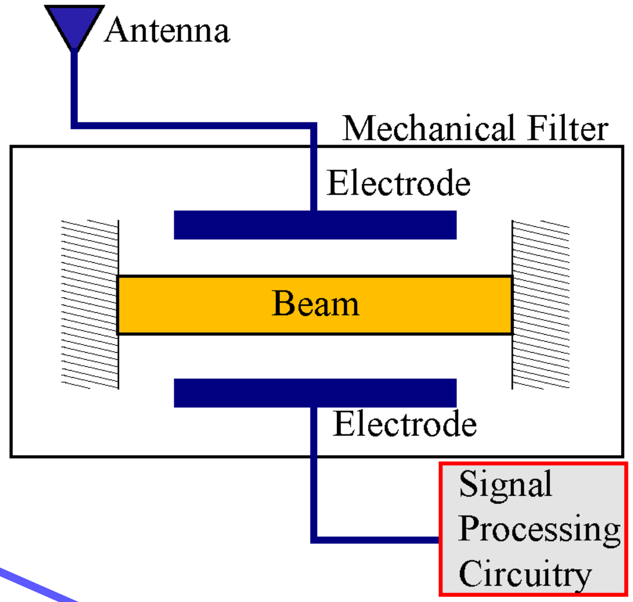
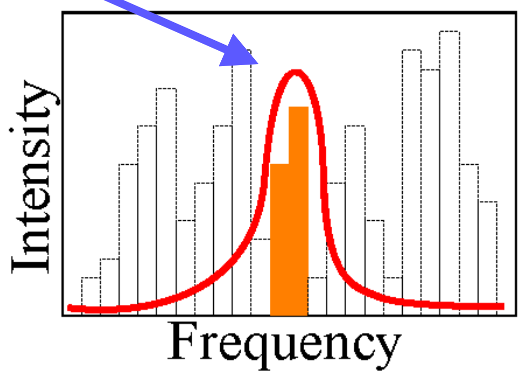
$$\approx Q^{-1}_{\text{anchor loss}} + Q^{-1}_{\text{ted}} + Q^{-1}_{\text{air}} + Q^{-1}_{\text{ohmic}} + Q^{-1}_{\text{other}}$$



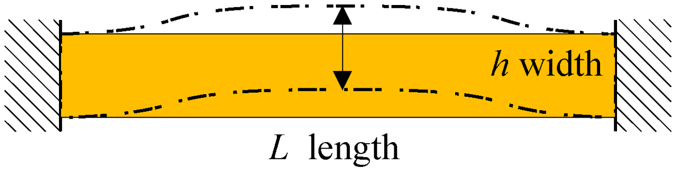


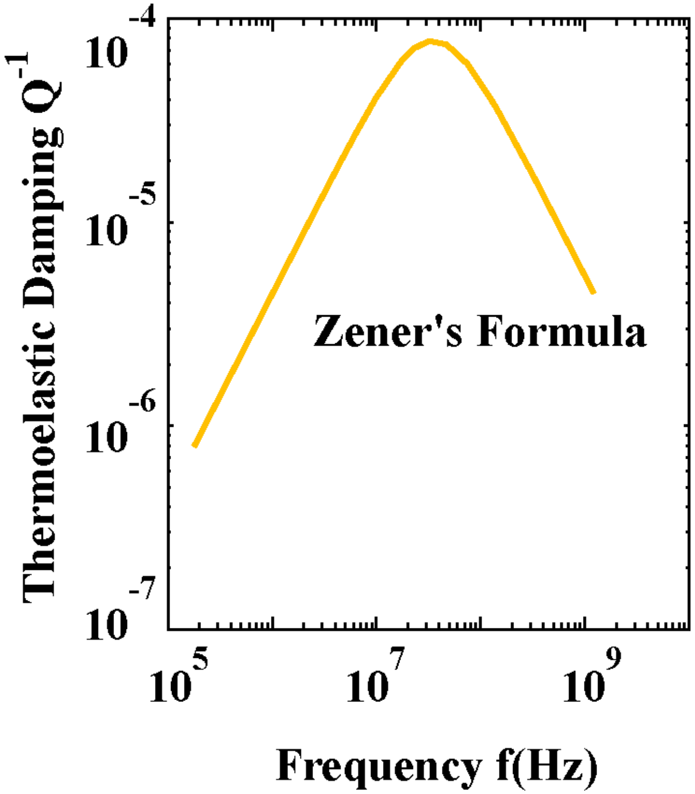
Zener's Model(1937)

- Obtained closed form algebraic relation for Q_{ted} of a beam.
- Experimentally verified.
- Based on Euler-Bernoulli beam theory

⬇

Applicable geometry is restricted.





C.Zener, Physical Review, 1937,pp230-235

Outline

- Governing equations (Non-dimensionalized form)
- Finite Element Discretization
- Time-harmonic response
  - Forced response
  - Modal response
- Closure

Governing Equations(Non-dimensionalized form)

- Equations obtained from thermodynamical principles
- Polysilicon is linear elastic, MEMS deformations are small.

→ -Assume linear elasticity.
-Temperature fluctuations are small
(Removes non-linearity in energy balance)

Balance of linear momentum

$$\ddot{\tilde{\mathbf{u}}} = \tilde{\boldsymbol{\nabla}} \cdot \left[\tilde{\mathbb{C}} : \tilde{\boldsymbol{\varepsilon}}\right] - \xi_1 \tilde{\boldsymbol{\nabla}} \tilde{\theta}$$

ξ_1: 4.6×10^{-7} — **Weak coupling**

Energy Balance

$$\dot{\tilde{\theta}} = \xi_2 \tilde{\boldsymbol{\nabla}}^2 \tilde{\theta} - \mathrm{tr}\left(\dot{\tilde{\boldsymbol{\varepsilon}}}\right)$$

ξ_2: 1.1×10^{-8} — **Strong coupling**

(PolySilicon values have been used for ξ.)

Finite Element Discretization

- Weak form

$$\boldsymbol{\sigma} = \mathbb{C} : \boldsymbol{\varepsilon} - \xi_1 \theta \mathbf{1}$$

$$\int_\Omega \ddot{\mathbf{u}} \cdot \mathbf{w} d\Omega + \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\mathbf{w}) d\Omega = \int_\Gamma (\boldsymbol{\sigma} \cdot \mathbf{n}) \cdot \mathbf{w} d\Gamma$$

$$\int_\Omega \dot{\theta} \delta\theta \, d\Omega + \xi_2 \int_\Omega \nabla \delta\theta \cdot \nabla \theta \, d\Omega + \int_\Omega \dot{\boldsymbol{\varepsilon}} : (\mathbb{C} : \mathbf{1}) \, \delta\theta \, d\Omega = \int_\Gamma (\nabla \theta) \cdot \mathbf{n} \, \delta\theta \, d\Gamma$$

- Discretized system of equations

$$\begin{bmatrix} \mathbf{M}_{uu} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \ddot{\mathbf{u}} \\ \ddot{\boldsymbol{\theta}} \end{pmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{C}_{\theta u} & \mathbf{C}_{\theta\theta} \end{bmatrix} \begin{pmatrix} \dot{\mathbf{u}} \\ \dot{\boldsymbol{\theta}} \end{pmatrix} + \begin{bmatrix} \mathbf{K}_{uu} & \xi_1 \mathbf{K}_{u\theta} \\ \mathbf{0} & \xi_2 \mathbf{K}_{\theta\theta} \end{bmatrix} \begin{pmatrix} \mathbf{u} \\ \boldsymbol{\theta} \end{pmatrix} = \begin{pmatrix} \mathbf{F}_u \\ \mathbf{F}_\theta \end{pmatrix}$$

1. M and C are singular, K is unsymmetric.

2. Diagonal blocks of M,C, and K are s.p.d.

3. Coupling term has relation, $\mathbf{C}_{\theta u} = -{\mathbf{K}_{u\theta}}^T$

Time-harmonic Response

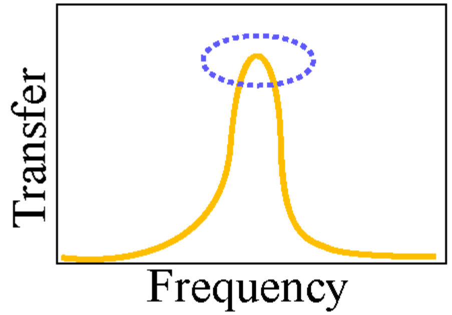
- Would like to evaluate: the Quality factor(Q)

➡ Assume time-harmonic response.

$$[\mathbf{u}, \boldsymbol{\theta}] = \left[\hat{\mathbf{u}}, \hat{\boldsymbol{\theta}}\right] e^{i\omega t}$$
$$\mathbf{F} = \hat{\mathbf{F}} e^{i\omega t}$$

Forced response → Evaluate transfer function Compute Q from peak.

$$\left(-\omega_{\text{force}}^2 \mathbf{M} + i\omega_{\text{force}} \mathbf{C} + \mathbf{K}\right) \begin{pmatrix} \hat{\mathbf{u}} \\ \hat{\boldsymbol{\theta}} \end{pmatrix} = \hat{\mathbf{F}}$$



Modal response

$$\left(-\omega^2 \mathbf{M} + i\omega \mathbf{C} + \mathbf{K}\right) \begin{pmatrix} \hat{\mathbf{u}} \\ \hat{\boldsymbol{\theta}} \end{pmatrix} = \mathbf{0}$$

$$Q = \frac{\text{Abs}(\omega)}{2\text{Im}(\omega)}$$

First-Order Form(GEP)

- By introducing an auxiliary variable we obtain a generalized eigenvalue problem(GEP).

Formulation where we obtain two Symmetric but Indefinite matrices

$$\mathbf{K}_{uu}\hat{\mathbf{v}} = i\omega \mathbf{K}_{uu}\hat{\mathbf{u}}$$

$$\begin{bmatrix} \mathbf{0} & -\mathbf{K}_{uu} & \mathbf{0} \\ -\mathbf{K}_{uu} & \mathbf{0} & -\xi_1 \mathbf{K}_{u\theta} \\ \mathbf{0} & \xi_1 \mathbf{C}_{\theta u} & \xi_1 \xi_2 \mathbf{K}_{\theta\theta} \end{bmatrix} \begin{pmatrix} \hat{\mathbf{u}} \\ \hat{\mathbf{v}} \\ \hat{\boldsymbol{\theta}} \end{pmatrix} = i\omega \begin{bmatrix} -\mathbf{K}_{uu} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{uu} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\xi_1 \mathbf{C}_{\theta\theta} \end{bmatrix} \begin{pmatrix} \hat{\mathbf{u}} \\ \hat{\mathbf{v}} \\ \hat{\boldsymbol{\theta}} \end{pmatrix}$$

Quadratic eigenvalue problem —Increases size→ GEP

$N = N_u + N_\theta$ → $2N_u + N_\theta$

67%(2D), 75%(3D) increase

Perturbation Method

- Exploit weak coupling in balance of linear momentum

0. Assume solution is equal to the mechanical problem plus a small perturbation.

$$\omega = \omega_0 + \delta\omega$$
$$\hat{\mathbf{u}} = \hat{\mathbf{u}}_0 + \delta\hat{\mathbf{u}}$$
$$\hat{\boldsymbol{\theta}} = \delta\hat{\boldsymbol{\theta}}$$

1. Solve for initial guess. — N_u GEP

$$\left(-\omega_0^2\mathbf{M}_{uu} + \mathbf{K}_{uu}\right)\hat{\mathbf{u}}_0 = \mathbf{0}$$

2. Compute corresponding thermal vector — N_θ linear solve

$$\left(i\omega_0\mathbf{C}_{\theta\theta} + \xi_2\mathbf{K}_{\theta\theta}\right)\hat{\boldsymbol{\theta}} = -i\omega_0\mathbf{C}_{\theta u}\hat{\mathbf{u}}_0$$

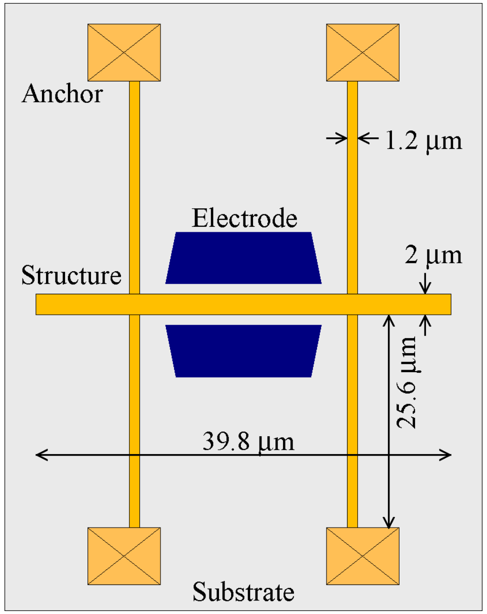
3. Solve for update by adding constraint $\hat{\mathbf{u}}_0^T\hat{\mathbf{u}} = \text{const}$ — N_u+1 linear solve

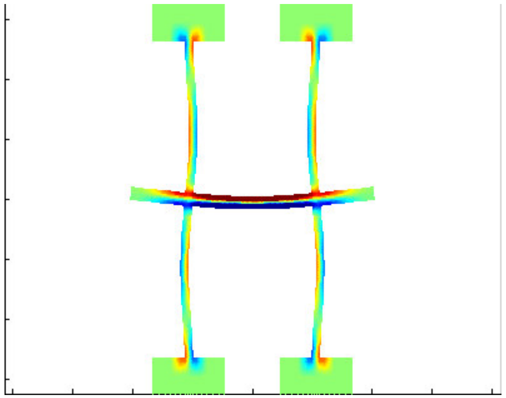
$$\begin{bmatrix} -\omega_0^2\mathbf{M}_{uu} + \mathbf{K}_{uu} & 2i\omega_0\mathbf{M}_{uu}\hat{\mathbf{u}}_0 \\ \hat{\mathbf{u}}_0^T & 0 \end{bmatrix}\begin{pmatrix} \delta\hat{\mathbf{u}} \\ i\delta\omega \end{pmatrix} = \begin{pmatrix} -\xi_1\mathbf{K}_{u\theta}\hat{\boldsymbol{\theta}} \\ 0 \end{pmatrix}$$

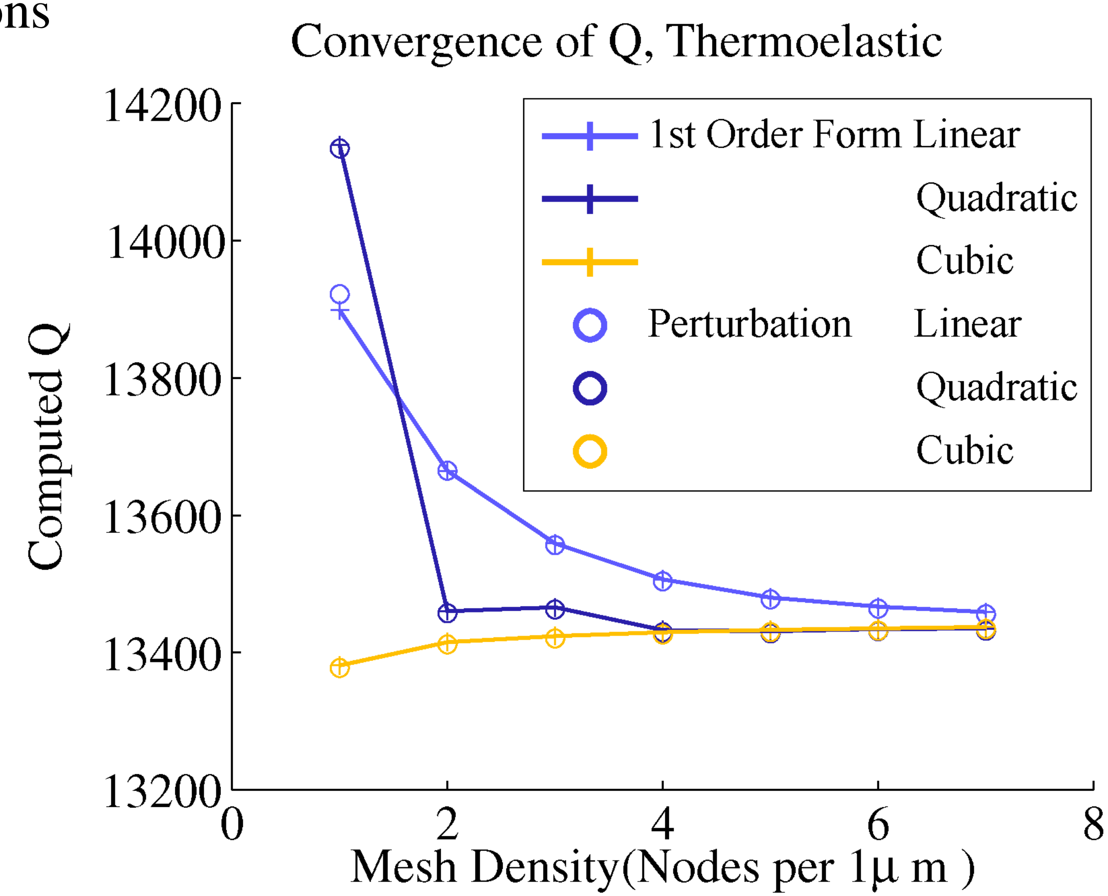
$2N_u+N_\theta$ GEP —Reduces size→ N_u GEP (Decreases to 27%(2D), 40%(3D)); N_θ, N_u+1 linear solve (Use LU from GEP)

Numerical example:Beam Structure

- 2D Plane Stress assumptions







$$Q_{\text{measured}} = 10743$$
$$Q_{\text{computed}} = 13423$$

W.-T.Hsu, J.R.Clark,C.T.-C.Nguyen,Transducers'01,pp1110-1113

ROM of the Forced Response

- Compute transfer function from forced response.

$$\left(-\omega_{\text{force}}^2 \begin{bmatrix} \mathbf{M}_{uu} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + i\omega_{\text{force}} \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{C}_{\theta u} & \mathbf{C}_{\theta\theta} \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{uu} & \xi_1 \mathbf{K}_{u\theta} \\ \mathbf{0} & \xi_2 \mathbf{K}_{\theta\theta} \end{bmatrix}\right) \begin{pmatrix} \hat{\mathbf{u}} \\ \hat{\boldsymbol{\theta}} \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{F}}_u \\ \hat{\mathbf{F}}_\theta \end{pmatrix}$$

Must solve size N system for each data point. Computationally expensive.

Compute a reduced order model, accurate around a center frequency, based on the Second Order Arnoldi method.

1. Generate sequence of vectors(spans 2nd Order Krylov subspace) which can describe the response.

$$\mathbf{V}_n = [\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n] \; (n << N) \qquad \begin{pmatrix} \hat{\mathbf{u}} \\ \hat{\boldsymbol{\theta}} \end{pmatrix} \approx \mathbf{V}_n \hat{\mathbf{g}}$$

2. Construct ROM by a Galerkin projection of the system onto the generated smaller subspace of solutions.

$$\left(-\omega_{\text{force}}^2 \mathbf{V}_n^* \mathbf{M} \mathbf{V}_n + i\omega_{\text{force}} \mathbf{V}_n^* \mathbf{C} \mathbf{V}_n + \mathbf{V}_n^* \mathbf{K} \mathbf{V}_n\right) \hat{\mathbf{g}} = \mathbf{V}_n^* \hat{\mathbf{F}}$$

SOAR for the Thermoelastic problem

- SOAR procedure(Bai and Su 2004)

$$
\begin{aligned}
\mathbf{r}_0 &= \mathbf{b}\left(=\hat{\mathbf{K}}^{-1}\hat{\mathbf{F}}\right) \\
\mathbf{r}_1 &= -\hat{\mathbf{K}}^{-1}\hat{\mathbf{C}}\mathbf{r}_0 \\
\mathbf{r}_j &= -\hat{\mathbf{K}}^{-1}\left(\hat{\mathbf{C}}\mathbf{r}_{j-1}+\hat{\mathbf{M}}\mathbf{r}_{j-2}\right)
\end{aligned}
$$

$$
\begin{aligned}
\hat{\mathbf{K}} &= \mathbf{K}+i\omega_{\text{center}}\mathbf{C}-\omega_{\text{center}}^2\mathbf{M} \\
\hat{\mathbf{C}} &= \mathbf{C}+2i\omega_{\text{center}}\mathbf{M}
\end{aligned}
$$

$$
\mathbf{V}_n = [\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n] = \begin{bmatrix}\mathbf{V}_n^u \\ \mathbf{V}_n^\theta\end{bmatrix}
$$

- Choice of basis for projection

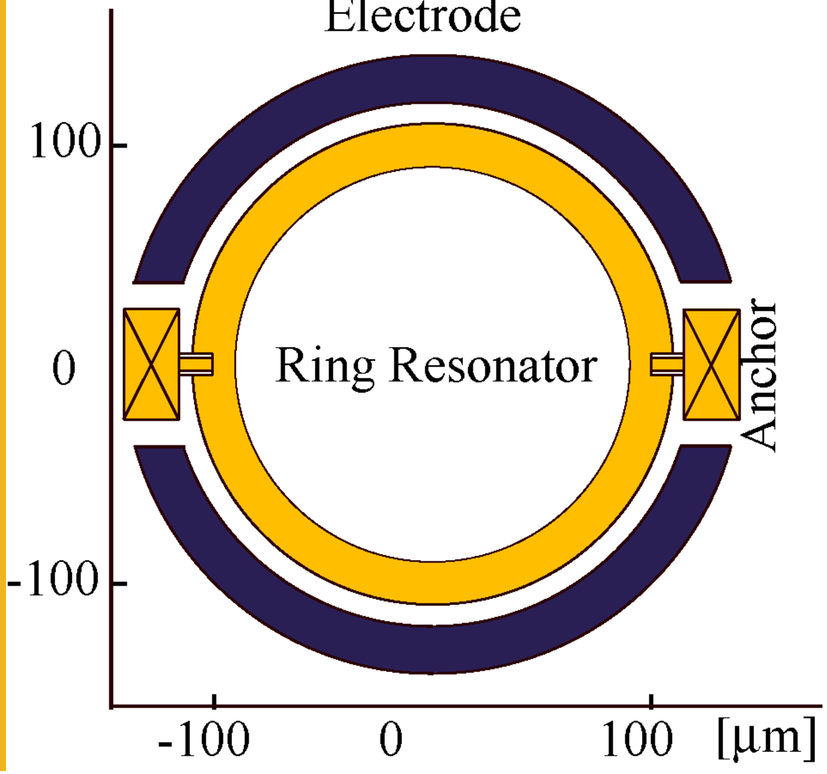
SOAR basis	M/T split basis
$\mathbf{V}_n$ (n)	$\mathbf{V}_n^{\text{split}} = \begin{bmatrix}\mathbf{V}_n^u & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_n^\theta\end{bmatrix}$ (2n) **Preserves matrix structure:** **1. h.p.d of diagonal submatrices** **2. zero structure**

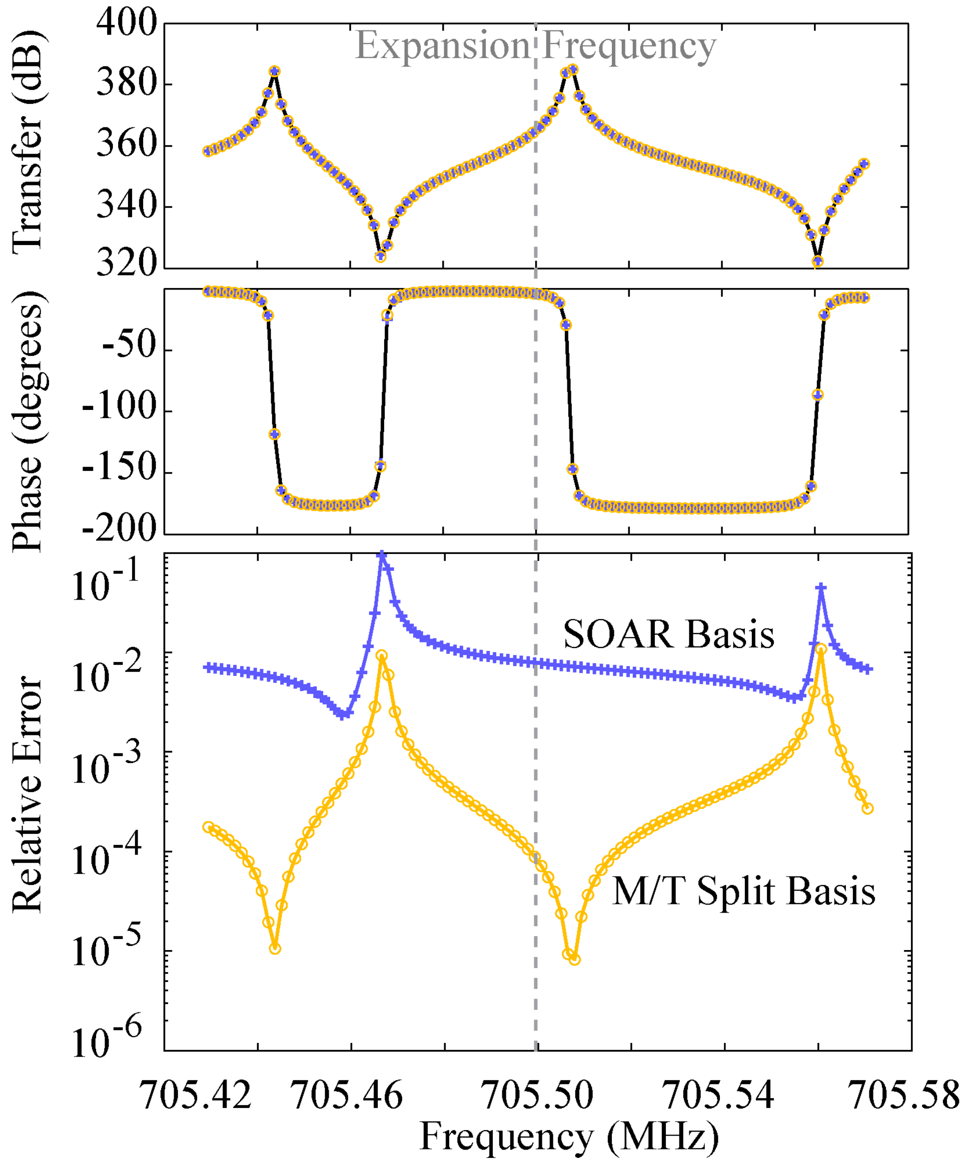
Z. Bai and Y. Su. SIAM J. Sci. Comp., 2004.

Numerical Example:Ring resonator

- 2D Plane Stress
- NDOF=38989
- ROM = 21

-Iterations: 20(SOAR Basis)
10(M/T Split Basis)

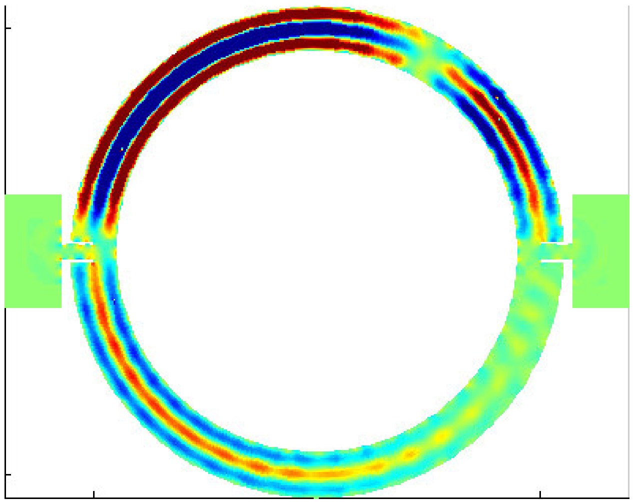




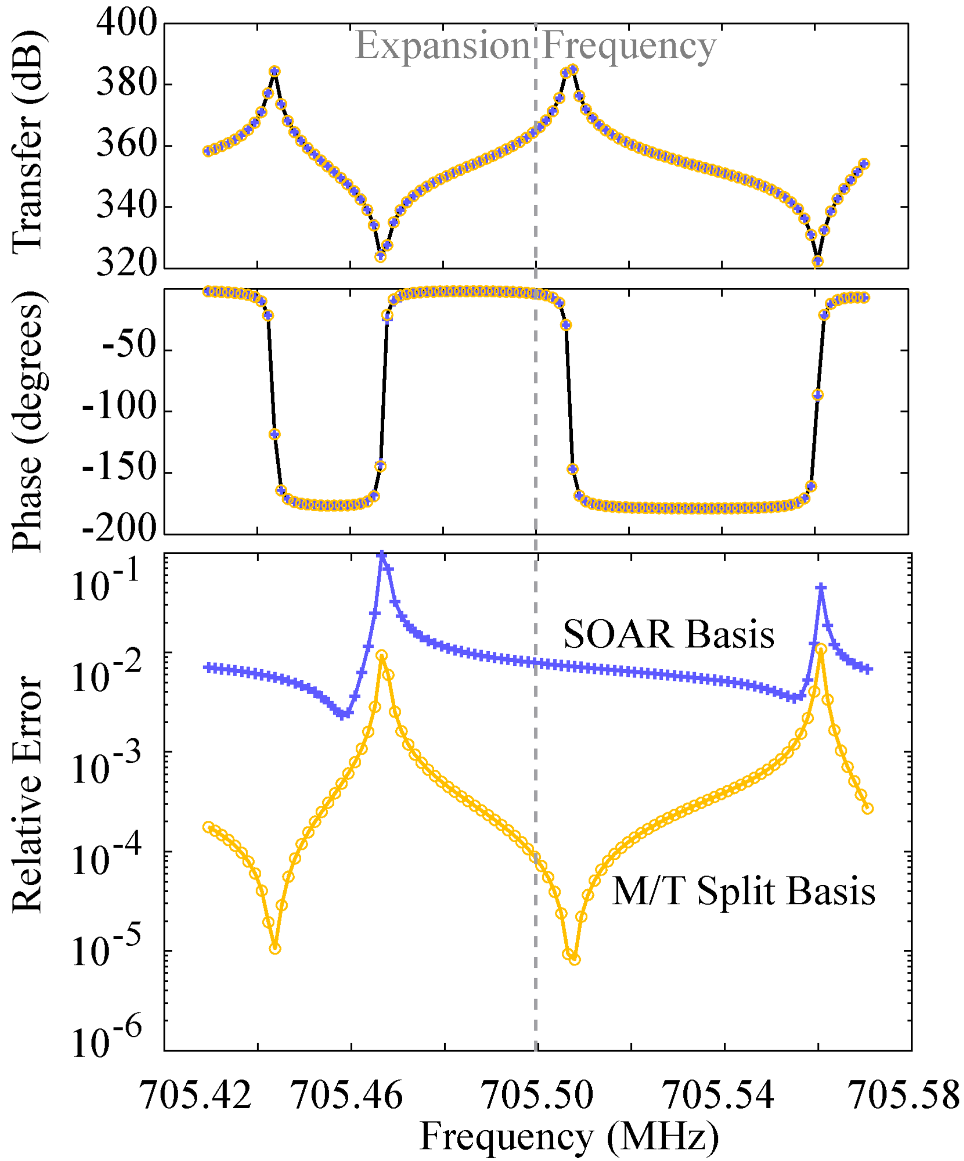
Numerical Example:Ring resonator

- 2D Plane Stress
- NDOF=38989 → **100 points Hours.**
- ROM = 21 → **Seconds.**

-Iterations: 20(SOAR Basis)
10(M/T Split Basis)



Forced at 705.5[MHz]



Closure

- Modal Response
  - Perturbative structure
  - $2N_u+N_\theta$ GEP → N_u GEP, N_θ, N_u+1 Linear solve
- Forced Response
  - SOAR N → n
  - M/T split basis → Preserves structure, Higher-order accuracy
- Extends to incorporate anchor loss with Perfectly Matched Layers(PML)

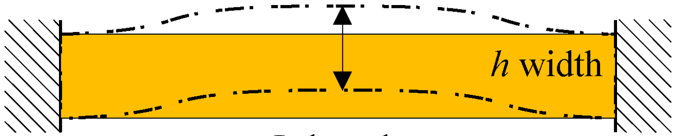
  ⇨ Complex symmetric matrices

Reference Slides

Zener's Model

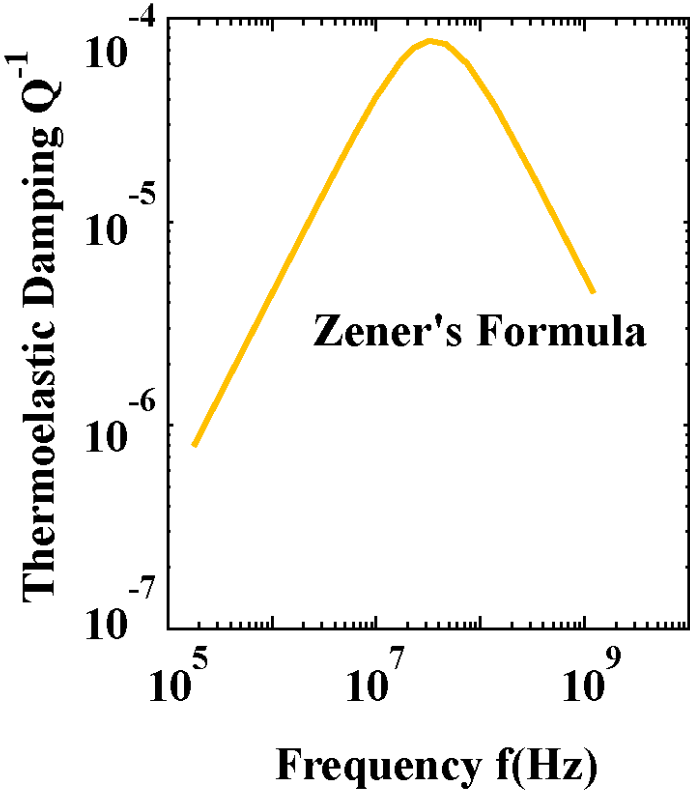
- Proposed by Zener in 1937.

-Evaluated Q of a beam in flexural mode



$$Q^{-1} = C\frac{\omega\tau}{1+(\omega\tau)^2}$$

$$\tau = \frac{\rho c_v h^2}{\pi^2 \kappa_T} \qquad C = \frac{\alpha_T T_0 E}{\rho c_v}$$

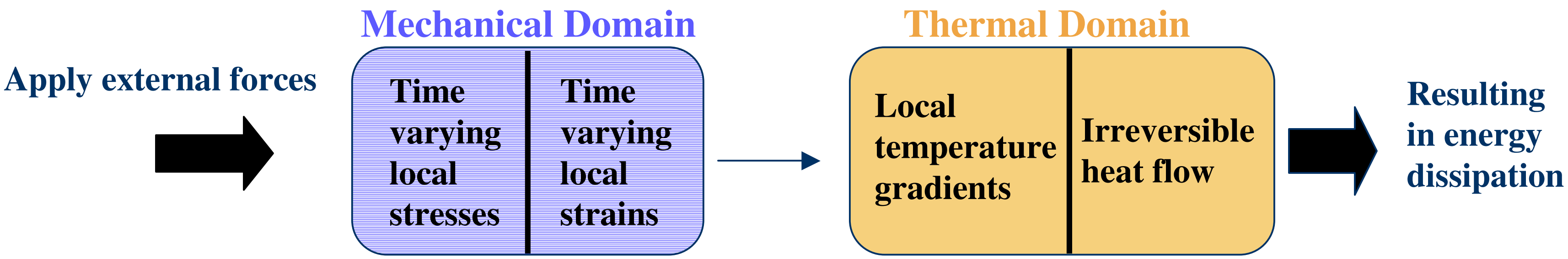


C.Zener, Physical Review, 1937,pp230-235

α_T : Linear coeffecient of thermal expansion
c_v : Specific heat (const. volume)
κ_T : Thermal conductivity
T_0 : Reference temperature
ρ : Density
E : Young's modulus
h : Beam width

Thermoelastic Damping

- Energy dissipation mechanism due to coupling between the mechanical and thermal domain.



Governing Equations

- Equations obtained from thermodynamical principles
- Polysilicon is linear elastic, MEMS deformations are small.

➡ -Assume linear elasticity.
-Temperature fluctuations are small
(Removes non-linearity in energy balance)

Balance of linear momentum

$$\rho \ddot{\mathbf{u}} = \boldsymbol{\nabla} \cdot [\mathbb{C} : \boldsymbol{\varepsilon}] \quad -3\kappa\alpha_T \boldsymbol{\nabla}\theta$$

Energy Balance

$$\rho c_v \dot{\theta} = \kappa_T \boldsymbol{\nabla}^2 \theta \qquad -3\kappa\alpha_T T_0 \,\mathrm{tr}(\dot{\boldsymbol{\varepsilon}})$$

Non-dimensionalization

- Expression for coefficients.

$$\ddot{\tilde{\mathbf{u}}} = \tilde{\boldsymbol{\nabla}} \cdot \left[\tilde{\mathbb{C}} : \tilde{\boldsymbol{\varepsilon}}\right] - \xi_1 \tilde{\boldsymbol{\nabla}} \tilde{\theta}$$

$$\dot{\tilde{\theta}} = \xi_2 \tilde{\boldsymbol{\nabla}}^2 \tilde{\theta} - \mathrm{tr}\left(\dot{\tilde{\boldsymbol{\varepsilon}}}\right)$$

ξ_1: 4.6×10^{-7}

ξ_2: 1.1×10^{-8}

$$\xi_1 = \frac{3\kappa\alpha_T^2}{\rho}\frac{T_0}{c_v} \qquad \xi_2 = \frac{\kappa_T}{c_v}\frac{1}{L\sqrt{\rho E}}$$

First-Order Form(GEP)

- By introducing an auxiliary variable we obtain a generalized eigenvalue problem(GEP).

1. $\hat{\mathbf{v}} = i\omega \mathbf{I}\hat{\mathbf{u}}$ --RHS matrix is s.p.d

$$\begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} \\ -\mathbf{K}_{uu} & \mathbf{0} & -\xi_1 \mathbf{K}_{u\theta} \\ \mathbf{0} & -\mathbf{C}_{\theta u} & -\xi_2 \mathbf{K}_{\theta\theta} \end{bmatrix} \begin{pmatrix} \hat{\mathbf{u}} \\ \hat{\mathbf{v}} \\ \hat{\boldsymbol{\theta}} \end{pmatrix} = i\omega \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{uu} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{C}_{\theta\theta} \end{bmatrix} \begin{pmatrix} \hat{\mathbf{u}} \\ \hat{\mathbf{v}} \\ \hat{\boldsymbol{\theta}} \end{pmatrix}$$

2. $\mathbf{K}_{uu}\hat{\mathbf{v}} = i\omega \mathbf{K}_{uu}\hat{\mathbf{u}}$ --Symmetric but indefinite matrices

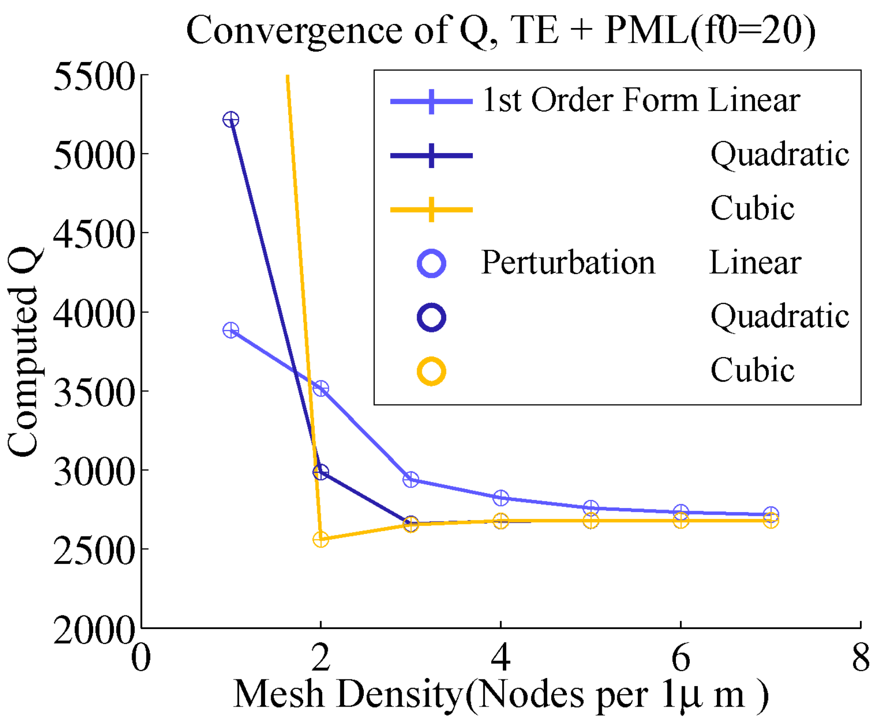
$$\begin{bmatrix} \mathbf{0} & -\mathbf{K}_{uu} & \mathbf{0} \\ -\mathbf{K}_{uu} & \mathbf{0} & -\xi_1 \mathbf{K}_{u\theta} \\ \mathbf{0} & \xi_1 \mathbf{C}_{\theta u} & \xi_1\xi_2 \mathbf{K}_{\theta\theta} \end{bmatrix} \begin{pmatrix} \hat{\mathbf{u}} \\ \hat{\mathbf{v}} \\ \hat{\boldsymbol{\theta}} \end{pmatrix} = i\omega \begin{bmatrix} -\mathbf{K}_{uu} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{uu} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\xi_1 \mathbf{C}_{\theta\theta} \end{bmatrix} \begin{pmatrix} \hat{\mathbf{u}} \\ \hat{\mathbf{v}} \\ \hat{\boldsymbol{\theta}} \end{pmatrix}$$

N_u+N_θ quadratic eigenvalue problem $\xrightarrow{\text{Increases size}}$ $2N_u+N_\theta$ GEP

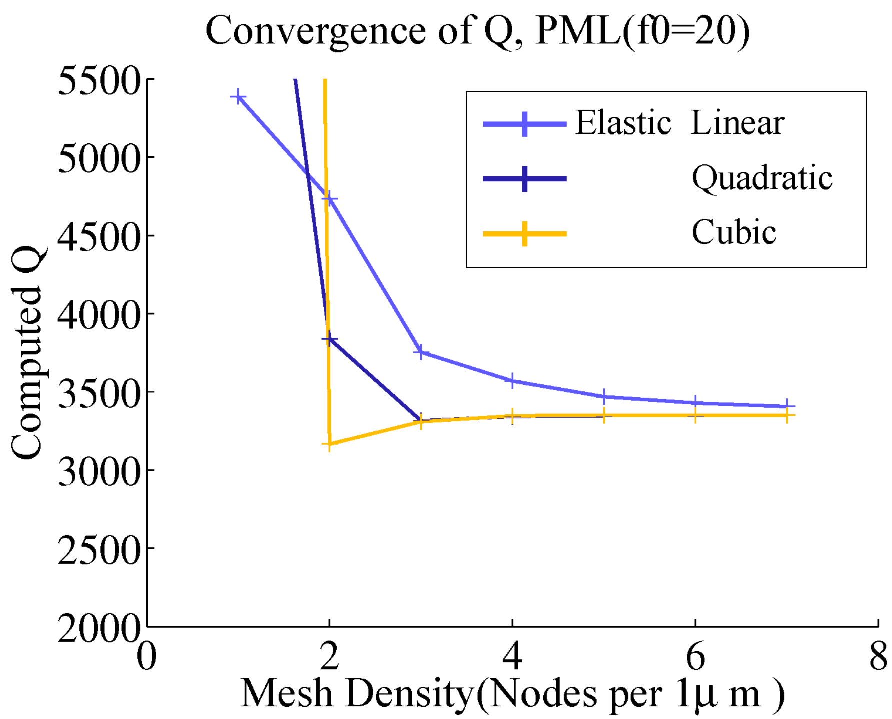
67%(2D), 75%(3D) increase

Example with PML:Michigan FF Beam

TED+PML



PML



$$Q_{\text{measured}} = 10743$$
$$Q_{\text{computed,ted}} = 13423$$
$$Q_{\text{computed,pml}} = 3350$$
$$Q_{\text{computed,ted+pml}} = 2682$$

Jacobi-Davidson Type of interation

The residuals of the eigenvalue problem are,

$$\begin{aligned} \mathbf{R}_u &= (\lambda^2 \mathbf{M}_{uu} + \mathbf{K}_{uu})\hat{\mathbf{u}} + \xi_1 \mathbf{K}_{u\theta}\hat{\boldsymbol{\theta}} \\ \mathbf{R}_\theta &= (\lambda \mathbf{C}_{\theta\theta} + \xi_2 \mathbf{K}_{\theta\theta})\hat{\boldsymbol{\theta}} + \lambda \mathbf{C}_{\theta u}\hat{\mathbf{u}} \end{aligned}$$

Then,

$$\begin{aligned} \delta\mathbf{R}_u &= (\lambda^2 \mathbf{M}_{uu} + \mathbf{K}_{uu})\delta\hat{\mathbf{u}} + \xi_1 \mathbf{K}_{u\theta}\delta\hat{\boldsymbol{\theta}} + (2\lambda \mathbf{M}_{uu}\hat{\mathbf{u}})\delta\lambda \\ \delta\mathbf{R}_\theta &= (\lambda \mathbf{C}_{\theta\theta} + \xi_2 \mathbf{K}_{\theta\theta})\delta\hat{\boldsymbol{\theta}} + \lambda \mathbf{C}_{\theta u}\delta\hat{\mathbf{u}} + (\mathbf{C}_{\theta\theta}\hat{\boldsymbol{\theta}} + \mathbf{C}_{\theta u}\hat{\mathbf{u}})\delta\lambda \end{aligned}$$

By adding a regularizing constraint on u,

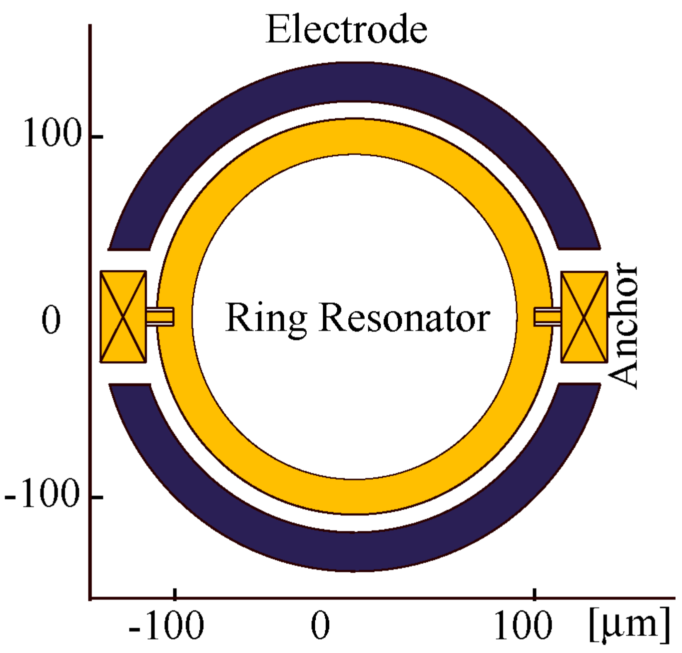
$$\left[\begin{array}{c} \delta\mathbf{R}_\theta \\ \hline \delta\mathbf{R}_u \\ \delta(\hat{\mathbf{u}}^T\hat{\mathbf{u}}) \end{array}\right] = \left[\begin{array}{c|cc} \lambda \mathbf{C}_{\theta\theta} + \xi_2 \mathbf{K}_{\theta\theta} & \lambda \mathbf{C}_{\theta u} & (\mathbf{C}_{\theta\theta}\hat{\boldsymbol{\theta}} + \mathbf{C}_{\theta u}\hat{\mathbf{u}}) \\ \hline \xi_1 \mathbf{K}_{u\theta} & (\lambda^2 \mathbf{M}_{uu} + \mathbf{K}_{uu}) & 2\lambda \mathbf{M}_{uu}\hat{\mathbf{u}} \\ 0 & \hat{\mathbf{u}}^T & 0 \end{array}\right] \left[\begin{array}{c} \delta\hat{\boldsymbol{\theta}} \\ \delta\hat{\mathbf{u}} \\ \delta\lambda \end{array}\right] = -\mathbf{R}$$

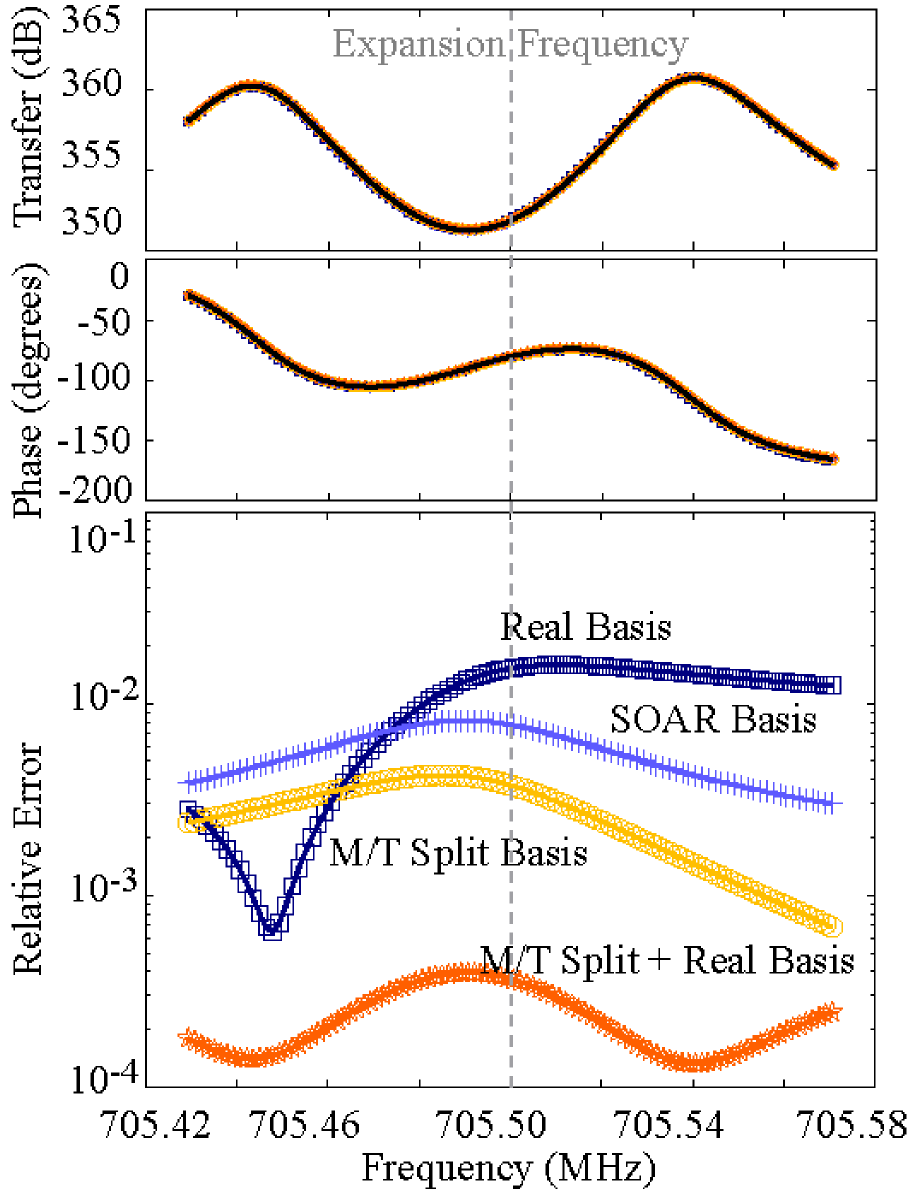
If we now drop the coupling term we obtain the Perturbation Method equations.
This iteration may be repeated to further refine the solution.

Example:Ring Resonator with PML

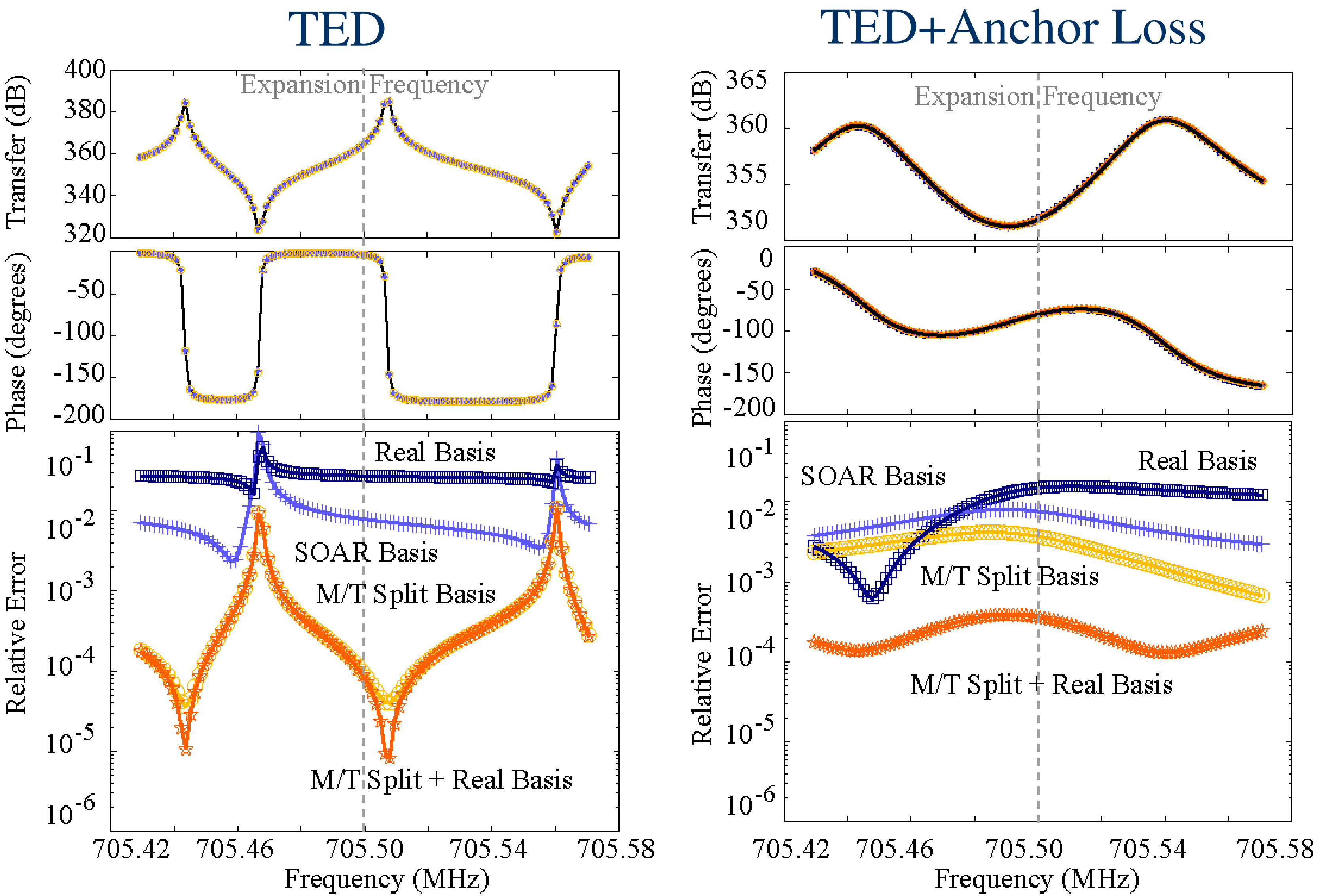
- 2D Plane Stress
- PML parameter f0=20
- NDOF=38989

ROM = 21 Normal n=20
M/T, Real n=10
M/T+Real n=5





TED vs. TED+PML(Ring Resonator)

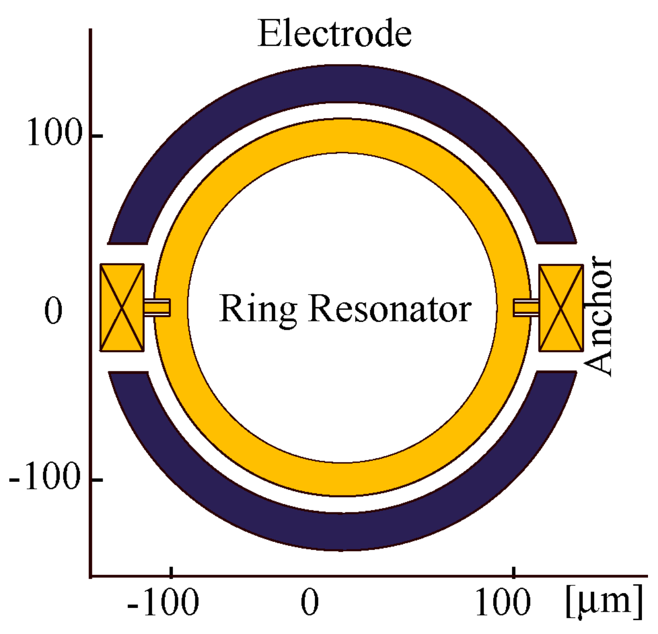


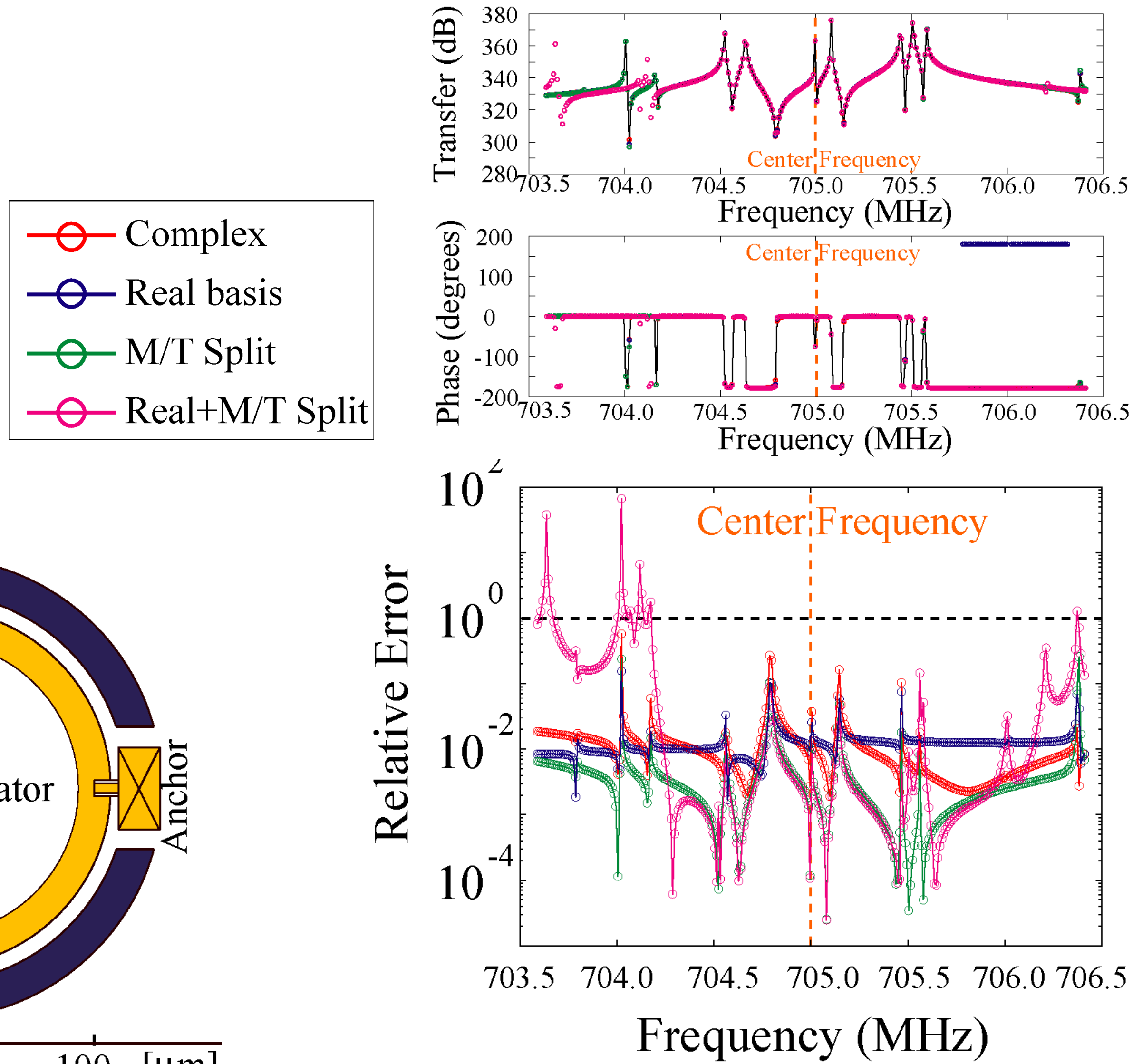
Numerical Example:Ring resonator

- 2D Plane Stress
- NDOF=38989

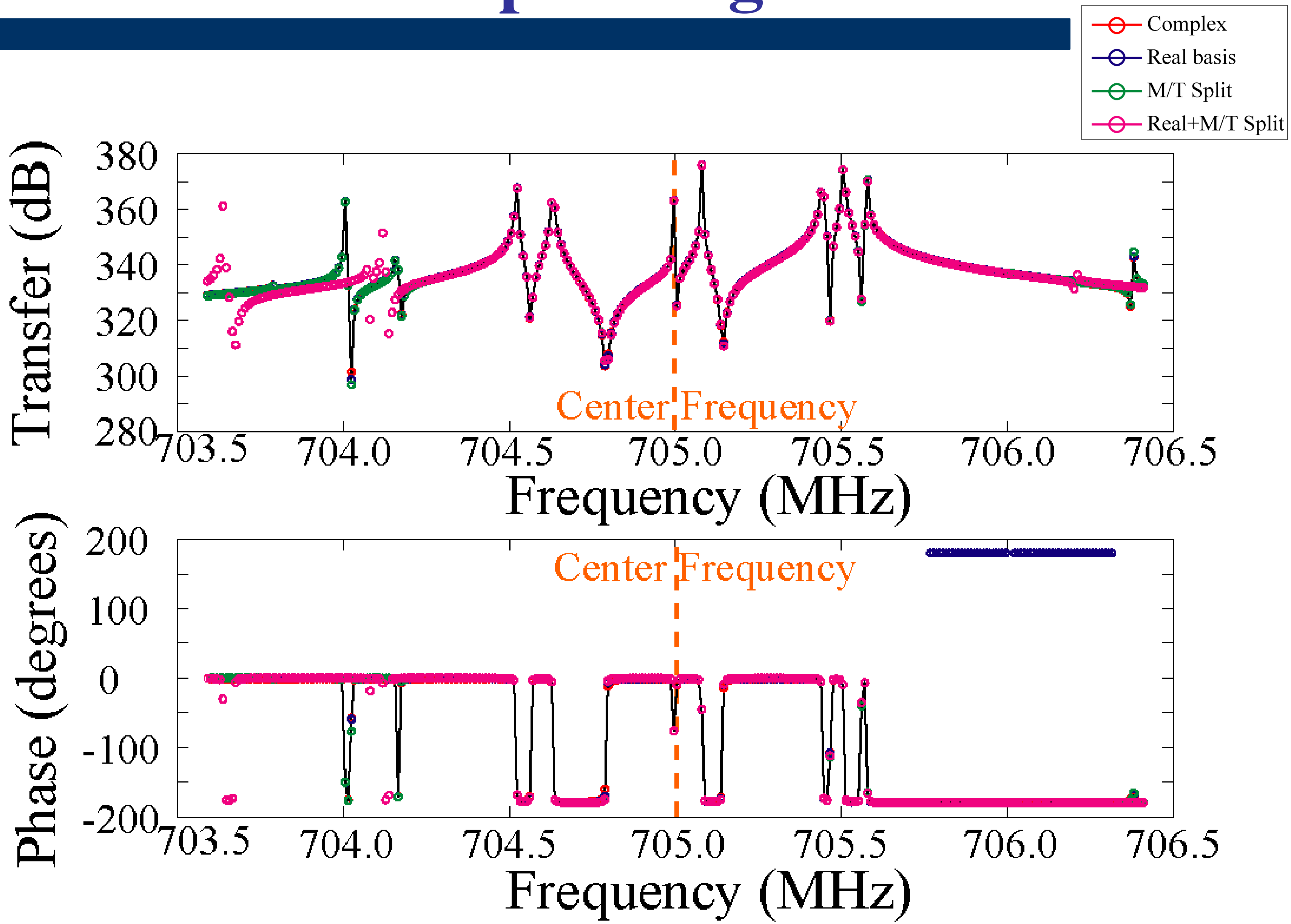
  ROM = 41

  Normal n=20
  M/T, Real n=10
  M/T+Real n=5

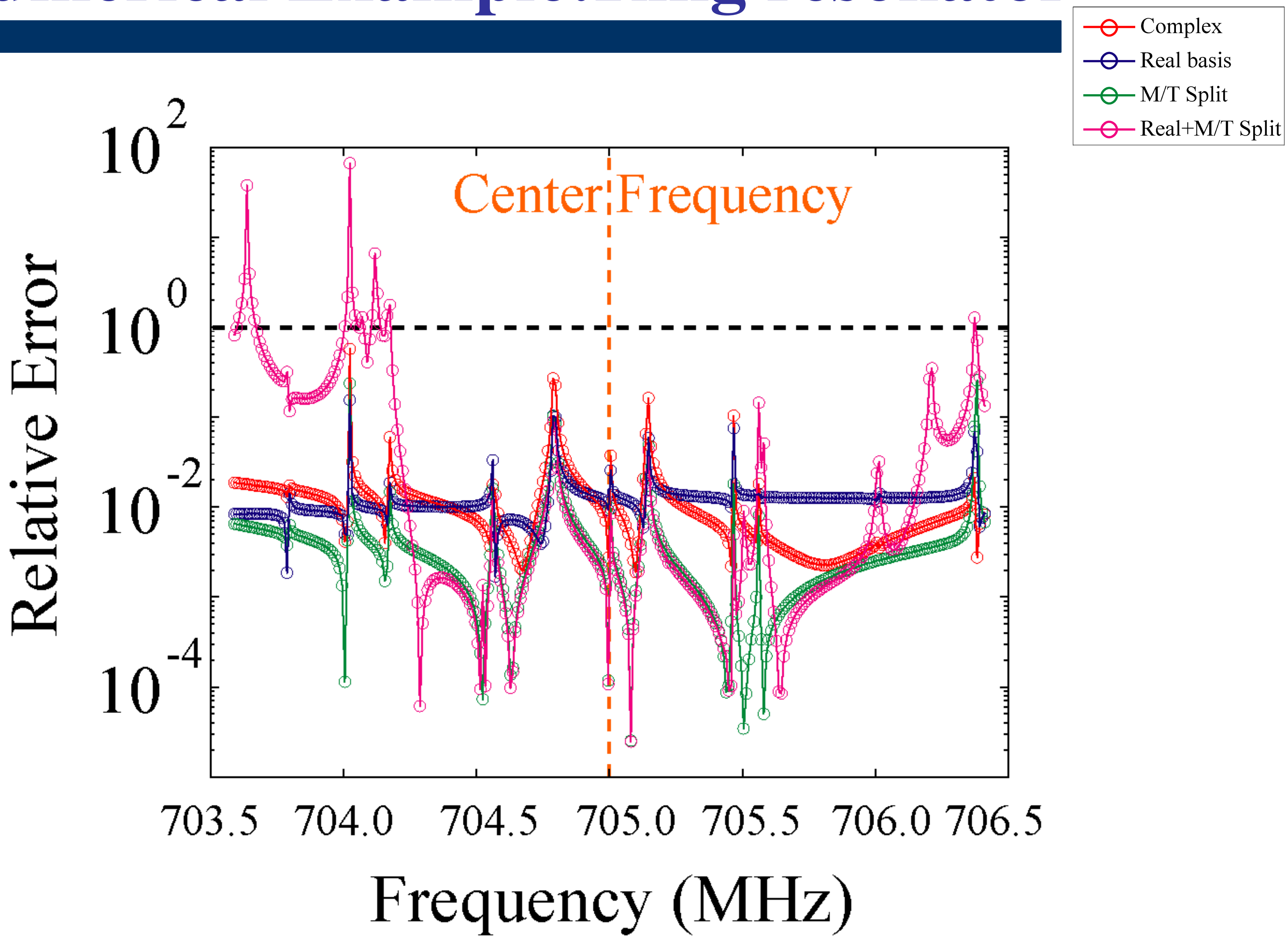




Numerical Example:Ring resonator



Numerical Example:Ring resonator



Choosing the Galerkin Projection Basis

- Choice of basis

$$\mathbf{V}_n = [\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n] = \begin{bmatrix} \mathbf{V}_n^u \\ \mathbf{V}_n^\theta \end{bmatrix}$$

	complex basis	real basis [diagonal submatrices remain s.p.d]
nonsplit basis	$\mathbf{V}_n$ (n)	$\mathbf{V}_n^{\mathrm{real}} = [\mathrm{Re}(\mathbf{V}_n), \mathrm{Im}(\mathbf{V}_n)]$ (2n) 2nd order accurate for pure mech. case
m/t split basis [matrix structure (zeros) preserved]	$\mathbf{V}_n^{\mathrm{split}} = \begin{bmatrix} \mathbf{V}_n^u & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_n^\theta \end{bmatrix}$ (2n) **Preserves** **1. s.p.d.(h.p.d.) of diagonal submatrices** **2. zeros in matrices**	$\mathbf{V}_n^{\mathrm{split,real}} = \left[\mathrm{Re}\left(\mathbf{V}_n^{\mathrm{split}}\right)\ \ \mathrm{Im}\left(\mathbf{V}_n^{\mathrm{split}}\right)\right]$ (4n) **Preserves** **1. complex symmetry of diagonal submatrices** **2. zeros in matrices**

Only TED (complex split basis) — **TED + Perfectly Matched Layer** (real split basis)