Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Solving generalized complex-symmetric eigenvalue problems arising from resonant MEMS simulations with PETSc

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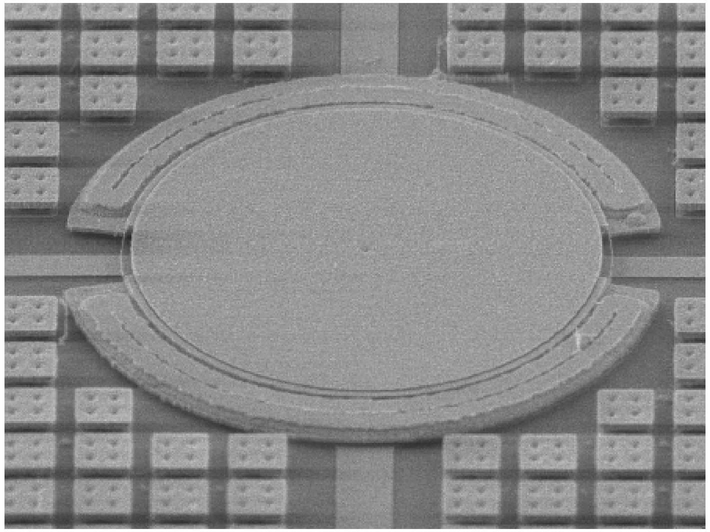
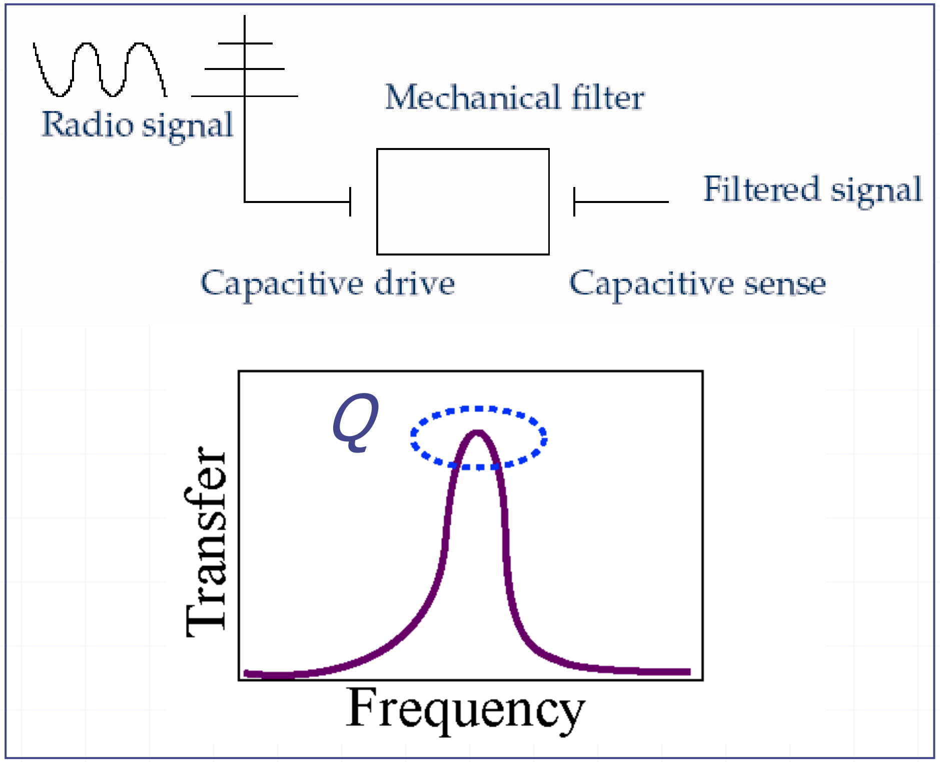
Institut für Mechanische Systeme

High-frequency MEMS resonators(MHz-GHz)

- Applications as small size, low energy consuming frequency references, filters, and sensors



SEM of 41.5 um radius poly-SiGe disk resonator



Resonator simulation

- Design requires knowledge of
  - Frequency
  - **Quality factor (Q)**

Existing Software and Methods → Frequency; ? → Quality factor

$$Q = \frac{\text{Maximum Stored Energy}}{\text{Energy Loss per radian}} \approx \frac{1}{\text{Damping}}$$

⇩

Tools and methods for evaluating damping in resonant MEMS

Mathematical problem

- Equation of motion discretized with FEM under harmonic assumption

Quadratic eigenvalue problem.

$$\left(-\omega^2 M + i\omega C + K\right)x = 0$$

Complex eigenfrequency

$$\omega = \omega_R + i\omega_I$$

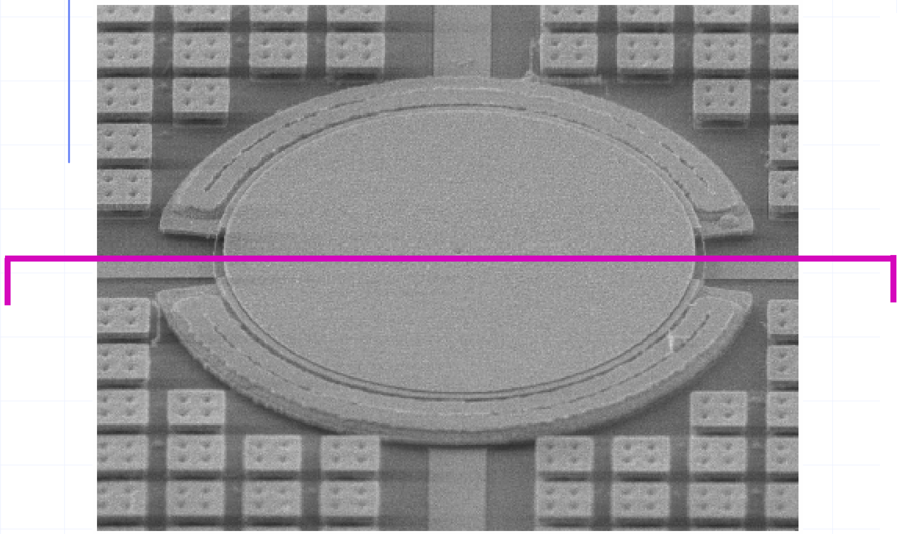
$$Q = \frac{|\omega|}{2\omega_I}$$

Overview

- Characteristics of the eigenvalue problem
  - Complex symmetry and interior eigenvalues
- Solution method
  - Projection methods
  - Jacobi-Davidson method (JDQZ)
  - Modifed multigrid for the correction equation (JDQZ)
- Software architecture
- Numerical example of a 2D beam
- Conclusions

Disk resonator (Anchor loss)

- Mechanism: Energy loss from radiating waves escaping into the substrate.



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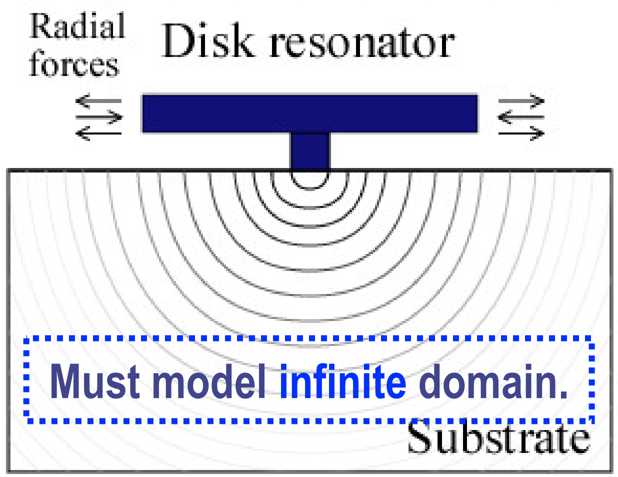
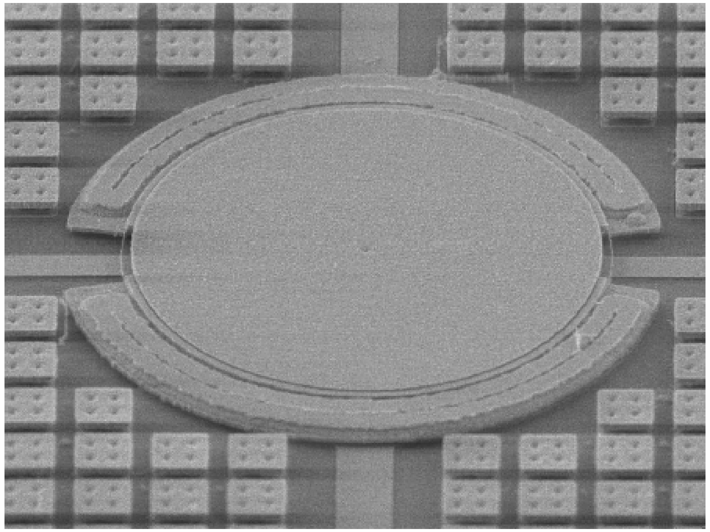


Section of disk resonator

Perfectly Matched Layers (PML)

- Mechanism: Energy loss from radiating waves escaping into the substrate.



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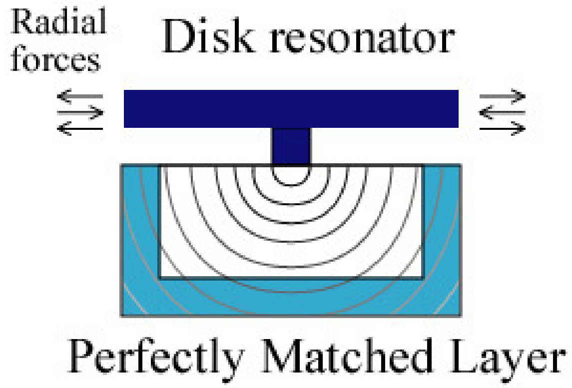
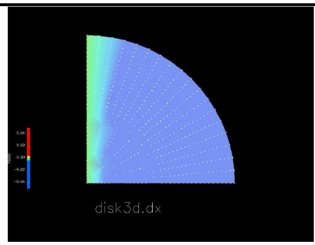
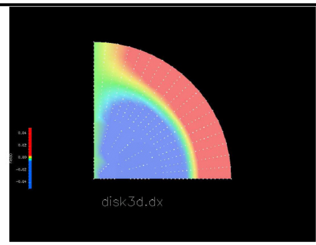
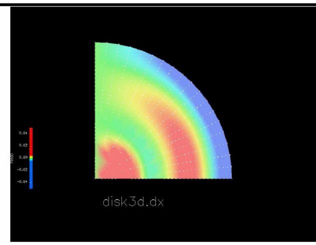
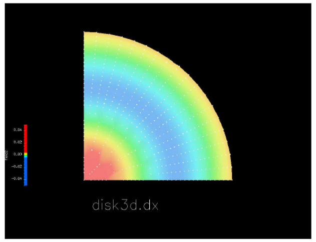
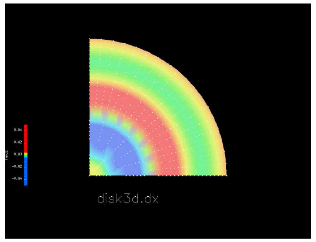
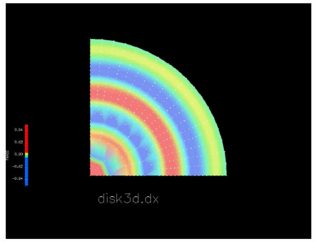
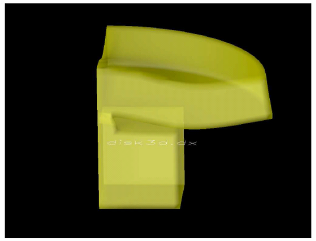
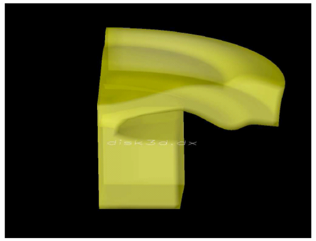
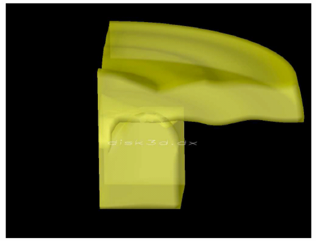


Outgoing waves are absorbed with zero impedance mismatch at PML boundaries.

Complex valued, Non-Hermitian matrices

Modes of vibration

Radial mode	1st	2nd	3rd
x disp.			
z disp.			
Shape			
Freq.[MHz]	289	759	1494
Nth mode	24	92	250-300

Difficulties

- PML implementation and high modes:

Generalized complex symmetric eigenvalue problem

$$\mathbf{Kx} = \omega^2 \mathbf{Mx}$$

1. Matrices are complex valued, Non-Hermitian

2. Desired modes are not the exterior of the spectrum but interior

Must deal with iterative methods for indefinite systems $\mathbf{K} - \omega^2 \mathbf{M}$

Projection methods

- Find eigen pair $(\omega^2, \mathbf{x} \in \mathcal{V})$ such that

$$\left(\mathbf{K} - \omega^2\mathbf{M}\right)\mathbf{x} \perp \mathcal{W}$$

- Galerkin projection(Ritz values,vectors) $\mathcal{W} = \mathcal{V}$
- Petrov-Galerkin projection $\mathcal{W} \neq \mathcal{V}$

Select harmonic-Ritz projection for better approximation of interior eigenvalues.

$$\mathcal{W} = \left(\mathbf{K} - \omega_0^2\mathbf{M}\right)\mathcal{V}$$

Jacobi-Davidson(JDQZ)

- Harmonic JD method for $(\mathbf{K}, \mathbf{M})$ target $\omega_0^2 = \frac{\alpha_0}{\beta_0}$

Given subspace $\mathcal{V}, \quad \mathcal{W} = (\beta_0 \mathbf{K} - \alpha_0 \mathbf{M})\, \mathcal{V}$

1. Find approximate eigenpair $((\alpha, \beta), \mathbf{u})$

$$\mathbf{u} \in \mathcal{V} \qquad (\beta \mathbf{K} - \alpha \mathbf{M})\, \mathbf{u} \perp \mathcal{W}$$

2. Solve correction equation for $\mathbf{t}$

3. Expand subspace

$$\mathcal{V}_{\text{new}} = \mathcal{V} \oplus \operatorname{span}\{\mathbf{t}\}$$
$$\mathcal{W}_{\text{new}} = \mathcal{W} \oplus \operatorname{span}\{(\beta_0 \mathbf{K} - \alpha_0 \mathbf{M})\, \mathbf{t}\}$$

(Fokkema, Sleijpen, and van der Horst, 1998)

Correction equation

$$(\mathbf{I}-\mathbf{p}\mathbf{p}^*)(\beta\mathbf{K}-\alpha\mathbf{M})(\mathbf{I}-\mathbf{u}\mathbf{u}^*)\,\mathbf{t}=-\mathbf{r}$$

$$\mathbf{r}=(\beta\mathbf{K}-\alpha\mathbf{M})\,\mathbf{u}$$
$$\mathbf{p}=(\beta_0\mathbf{K}-\alpha_0\mathbf{M})\,\mathbf{u}\in\mathcal{W}$$

- Accurate solve is not required.

  For large scale problems, iterative methods such as GMRES with Multigrid preconditioning is known to be effective for mechanical problems.

Use Smooth Aggregation Multigrid

⇨ $$\mathbf{P}_{\mathrm{MG}}\approx(\beta_0\mathbf{K}-\alpha_0\mathbf{M})$$

(Vanek, Mandel, and Brezina, 1995)

Correction equation

$$(\mathbf{I} - \mathbf{p}\mathbf{p}^*)\mathbf{P}_{\mathrm{MG}}(\mathbf{I} - \mathbf{u}\mathbf{u}^*)\,\mathbf{t} = -\mathbf{r}$$

$$\mathbf{r} = (\beta\mathbf{K} - \alpha\mathbf{M})\,\mathbf{u}$$
$$\mathbf{p} = (\beta_0\mathbf{K} - \alpha_0\mathbf{M})\,\mathbf{u} \in \mathcal{W}$$

- Accurate solve is not required.

  For large scale problems, iterative methods such as GMRES with Multigrid preconditioning is known to be effective for mechanical problems.

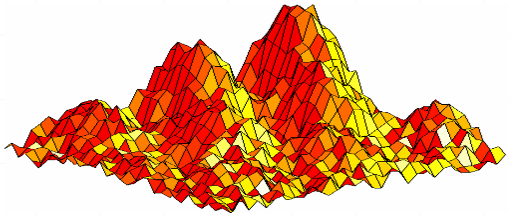
Use Smooth Aggregation Multigrid

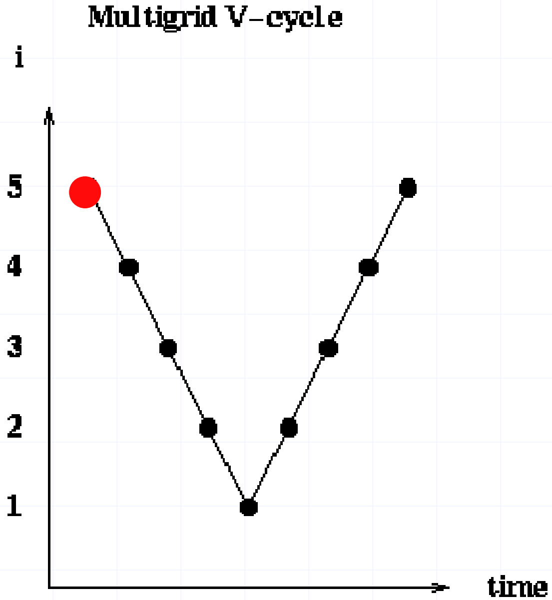
⇨ $$\mathbf{P}_{\mathrm{MG}} \approx (\beta_0\mathbf{K} - \alpha_0\mathbf{M})$$ Not S.P.D.

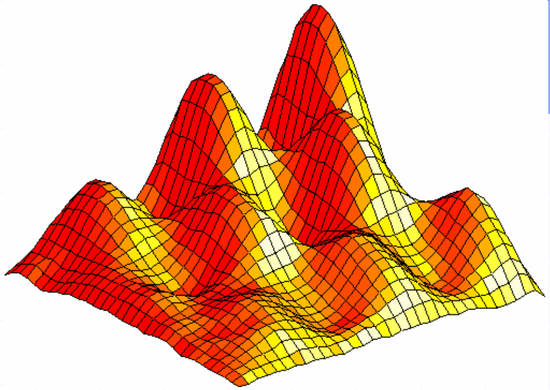
(Vanek, Mandel, and Brezina, 1995)

Algebraic Multigrid

- Smoothed aggregation algebraic multigrid



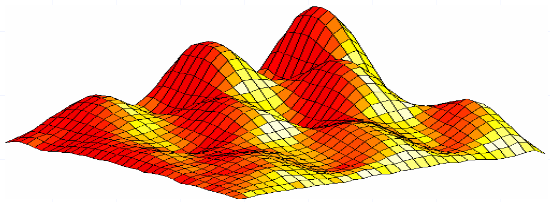




(Demmel, Parallel Computing Notes)

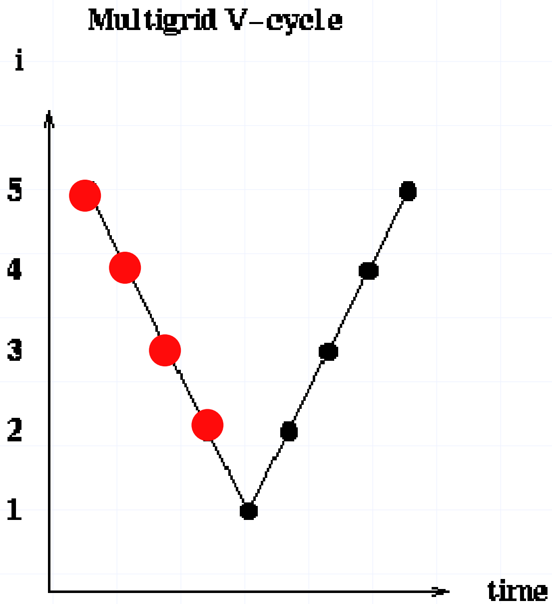
Algebraic Multigrid

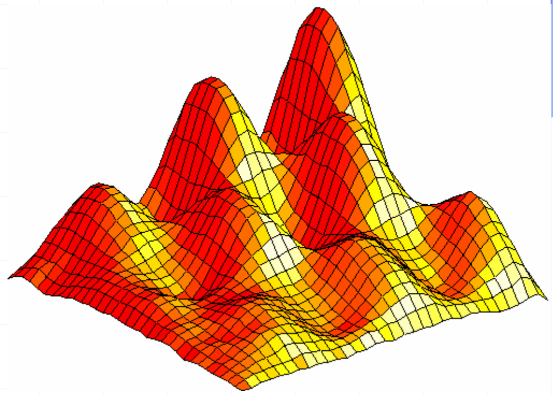
- Smoothed aggregation algebraic multigrid



Smoother

Remove error with large eigenvalue





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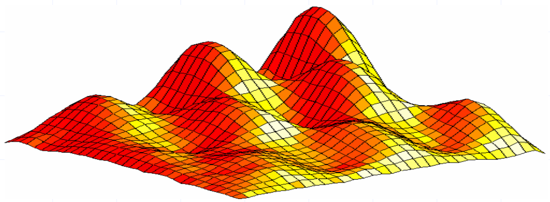

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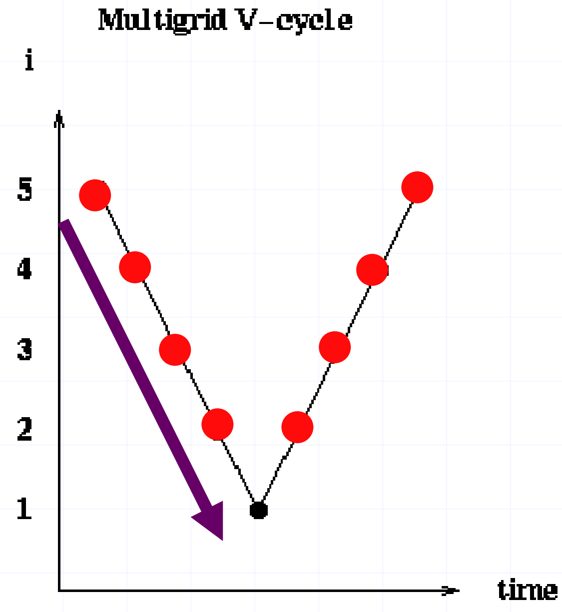

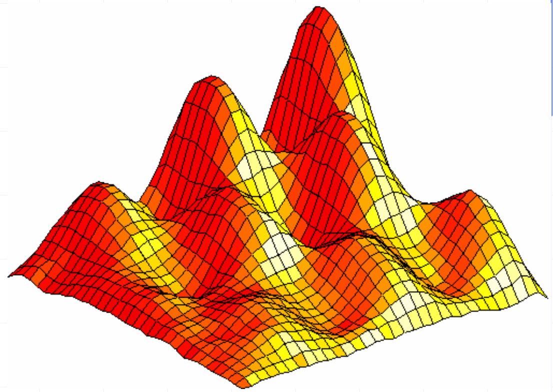

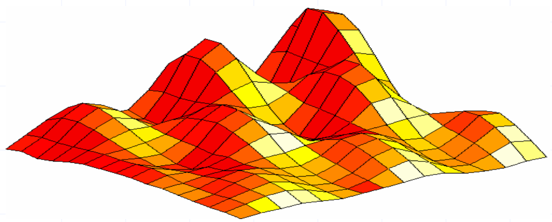

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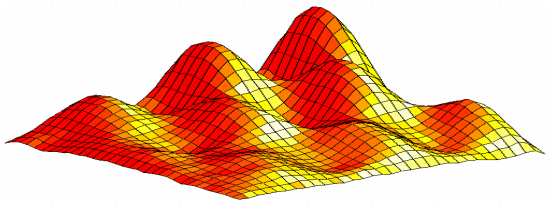

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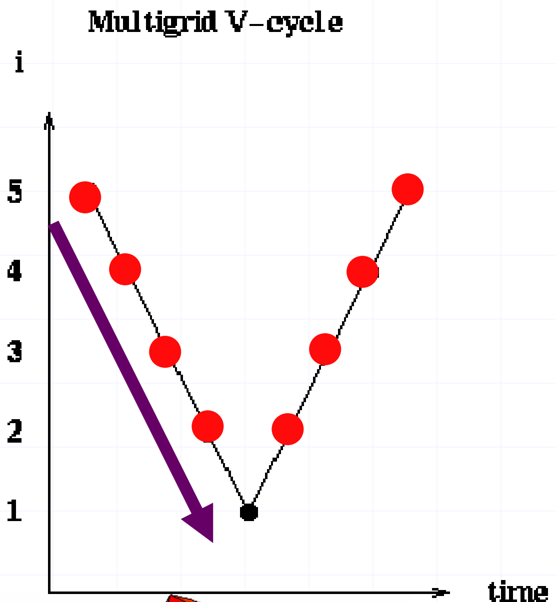

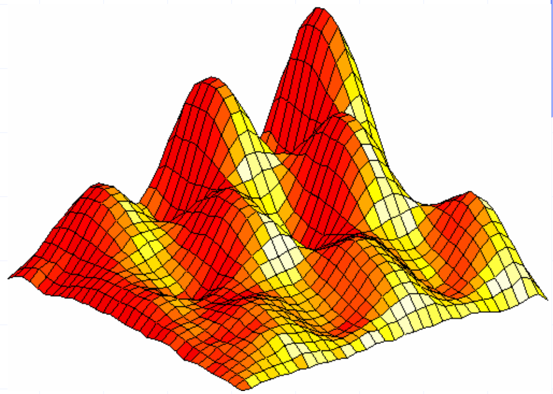

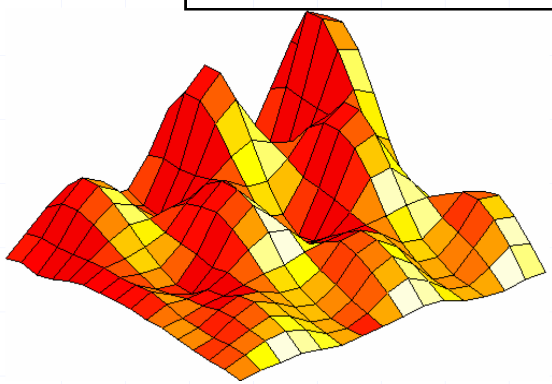

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Algebraic Multigrid

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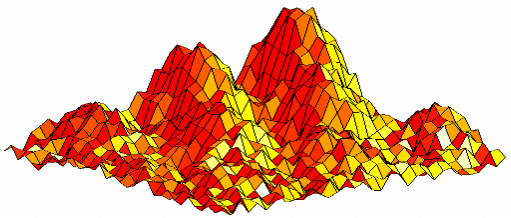
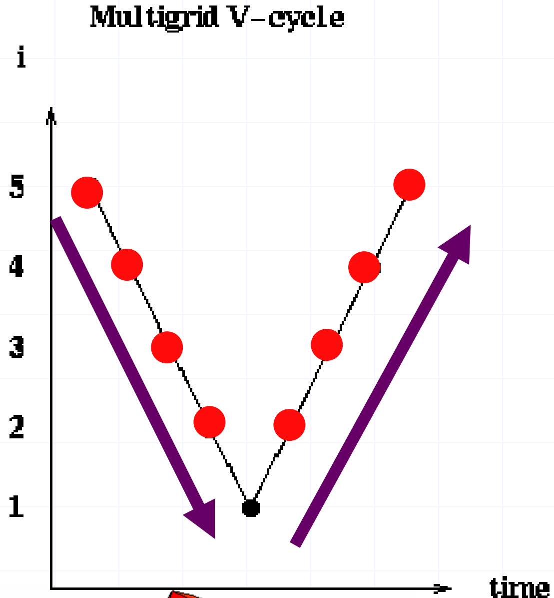
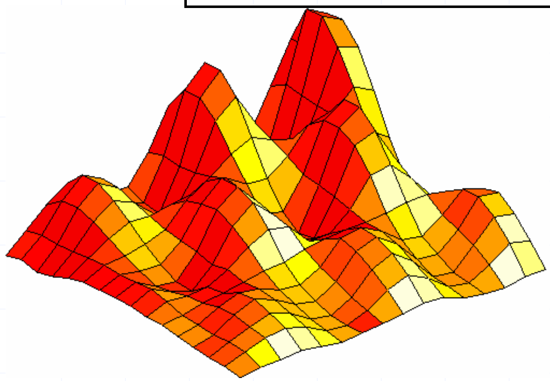
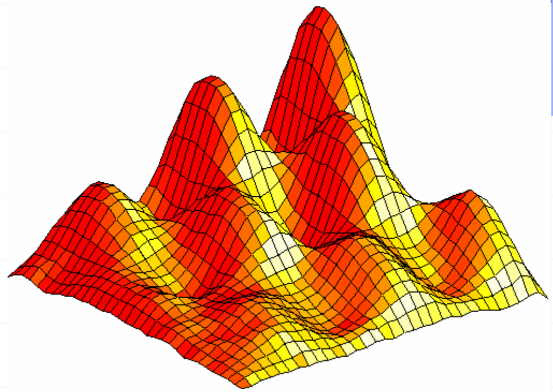


Smoother

Remove error with large eigenvalue







Coarse grid correction (Prollongator)

Remove error with small eigenvalue

(Demmel, Parallel Computing Notes)

Smother for $\mathbf{Ax} = \mathbf{b}$

- Gauss-Seidel

  Does not converge and smooth for indefinite case.

- Row projection method (Kaczmarz, 1937)

Gauss-Seidel $\boxed{\mathbf{AA}^*\mathbf{y} = \mathbf{b}}$

$$\mathbf{A}^*\mathbf{y} = \mathbf{x}$$

Converges if $\mathbf{b} \in \mathrm{range}(\mathbf{A})$(Tanabe, 1971)

Parallel version (Gordan and Gordon, 2005)

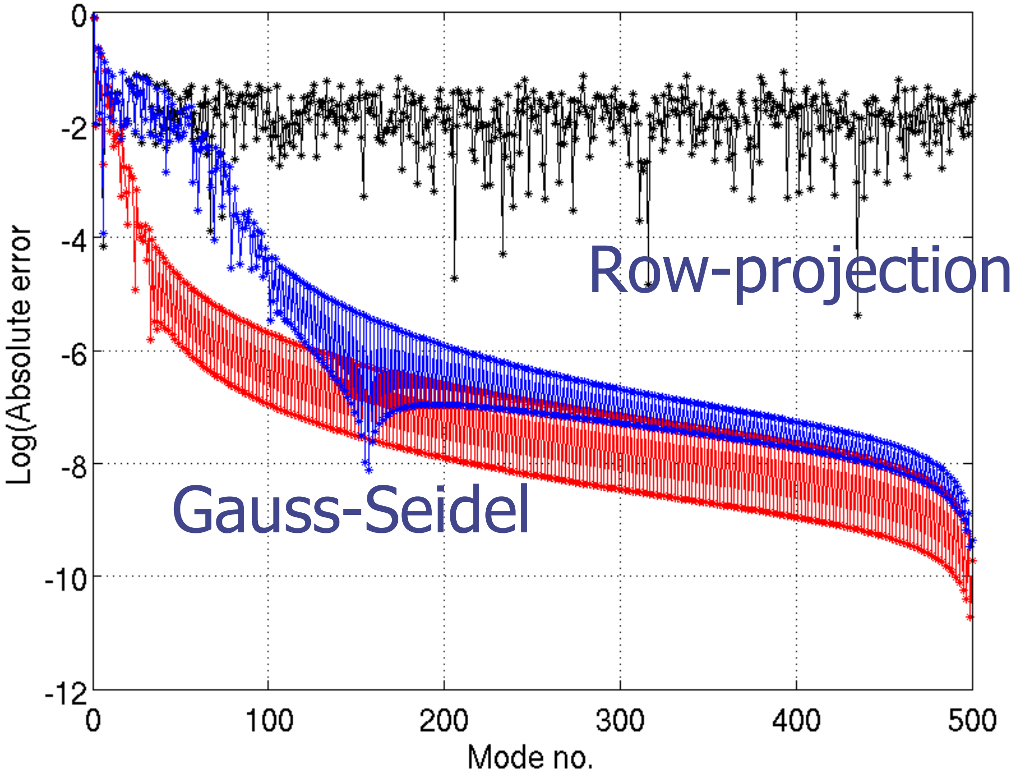
Smoother performance (SPD)

- 1-D Helmholtz fixed boundaries

$$\mathbf{A} = \mathrm{tridiag}[-1, 2, -1]$$

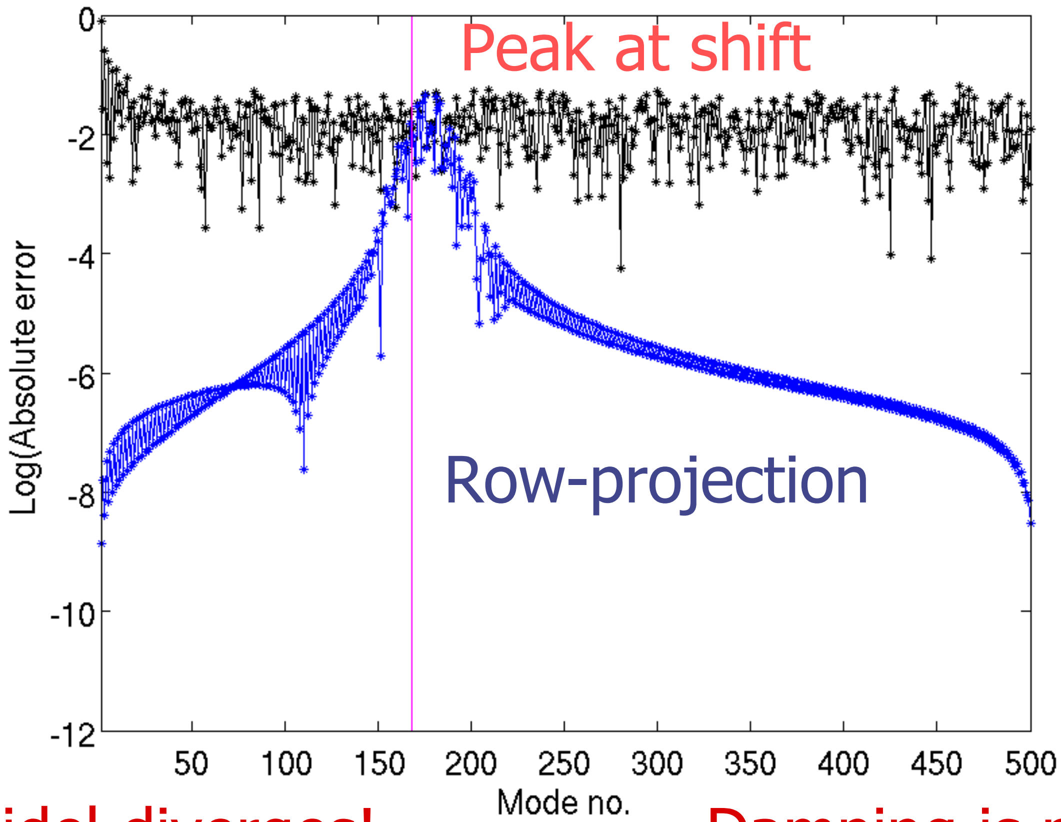
$$\mathbf{Ax} = 0$$

$$\mathbf{x} = \mathrm{rand}$$



Smoother performance (indefinite)

- Shift into ¼ of the spectrum $(\mathbf{A} - \sigma \mathbf{I})\,\mathbf{x} = 0$



Gauss-Seidel diverges!

Damping is not superior but does work!

Prolongators

- Vectors based on near null space of operator $\mathbf{K}$

$$\mathbf{V}_{\mathrm{Null}} = [\ \mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_N\ ]$$

Real to complex formulation

$$\mathbf{A} := \mathbf{K} - \omega_0^2\mathbf{M} = \mathbf{A}_r + i\mathbf{A}_i$$

$$\widehat{\mathbf{A}} = \begin{bmatrix} \mathbf{A}_r & -\mathbf{A}_i \\ \mathbf{A}_i & \mathbf{A}_r \end{bmatrix}$$

$$\widehat{\mathbf{V}}_{\mathrm{Null}} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{0} & \mathbf{v}_2 & \mathbf{0} & \cdots & \mathbf{v}_N & \mathbf{0} \\ \mathbf{0} & \mathbf{v}_1 & \mathbf{0} & \mathbf{v}_2 & \cdots & \mathbf{0} & \mathbf{v}_N \end{bmatrix}$$

Hybrid software architecture

- Petsc

  Fast implementation

  - Linear algebra objects(vector, matrices), solvers, and related operations.

- Trilinos

  User friendly architecture

  - ML(MultiLevel) package for algebraic multigrid.

- HiQLab

  Flexibility

  - Generates the mass, stiffness matrix data.

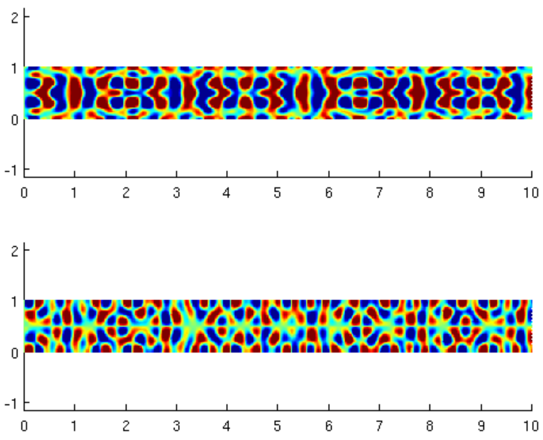
Symmetric indefinite problem

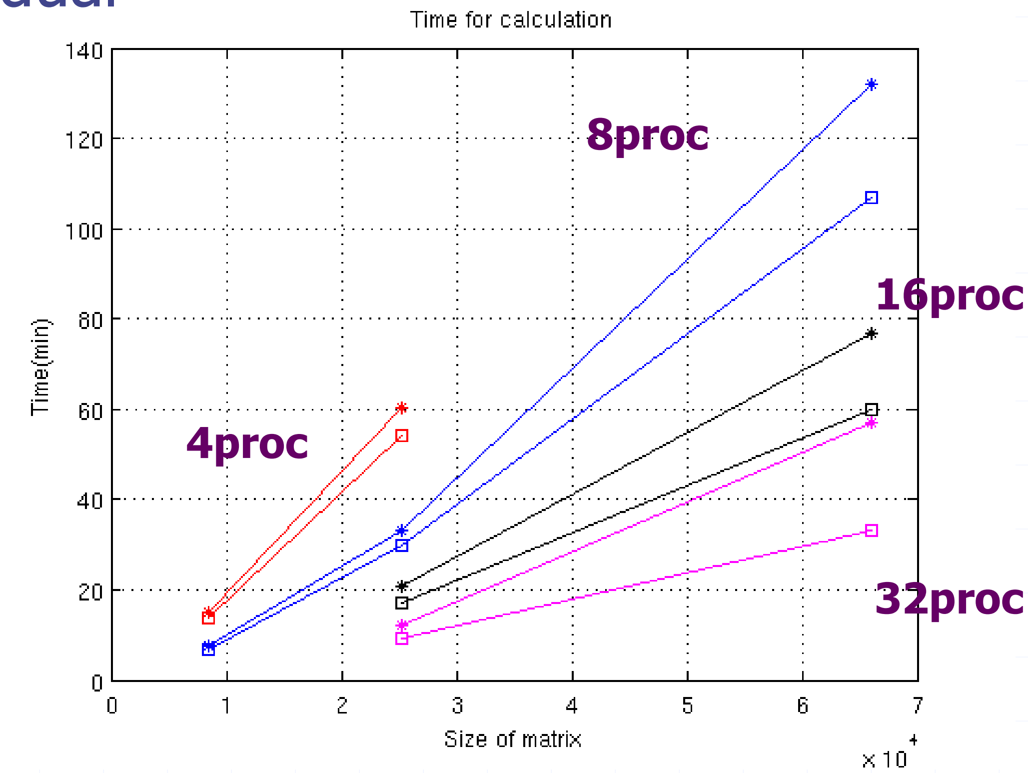
- 2D elastic beam

$$\mathbf{P}_{\mathrm{MG}} \approx (\beta_0 \mathbf{K} - \alpha_0 \mathbf{M})$$

s.p.d

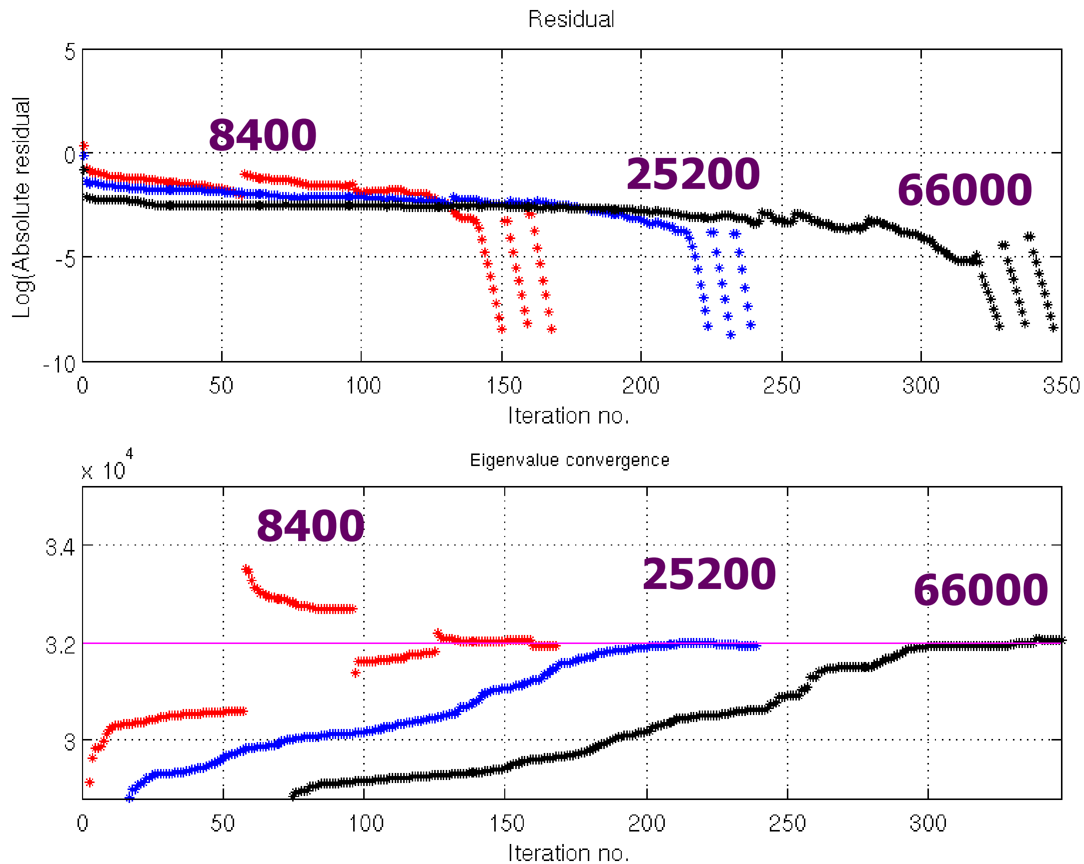
Search for 3 eigenvalues near 500th from bottom with residual tol 1e-8.





Convergence behavior

- Comparison of 8 proc for different discretizations.



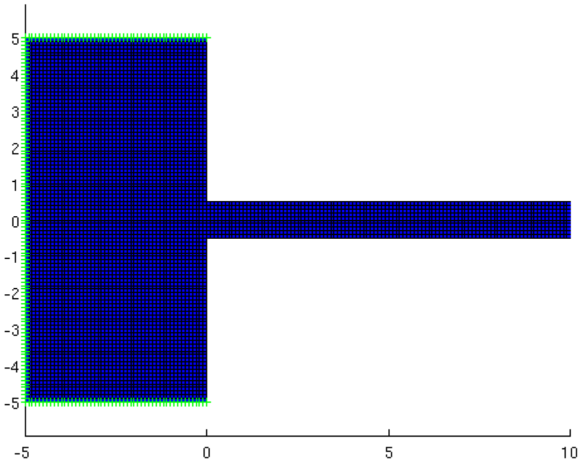
Complex symmetric problem

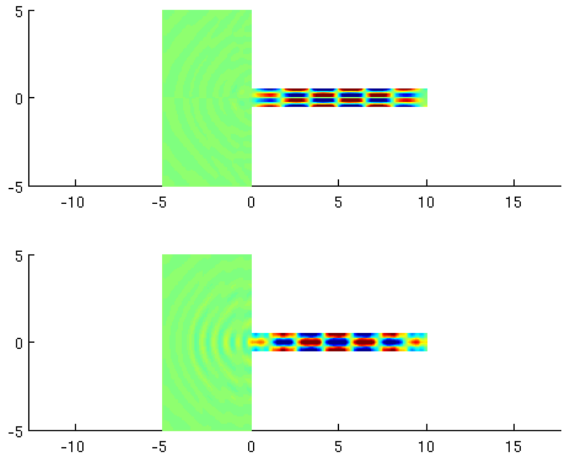
- 2D elastic beam with PML

$$\mathbf{P}_{\mathrm{MG}} \approx (\beta_0 \mathbf{K} - \alpha_0 \mathbf{M})$$

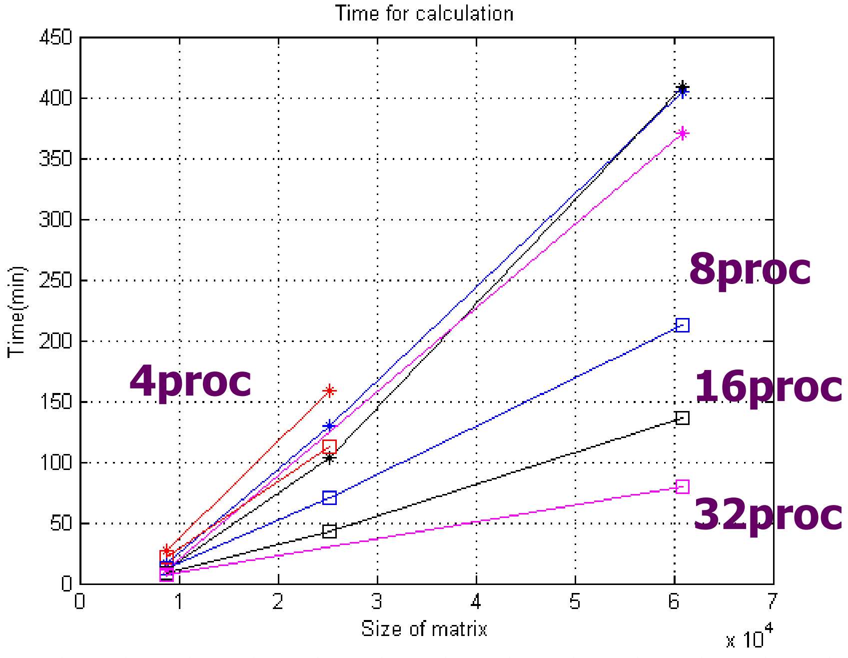
complex sym

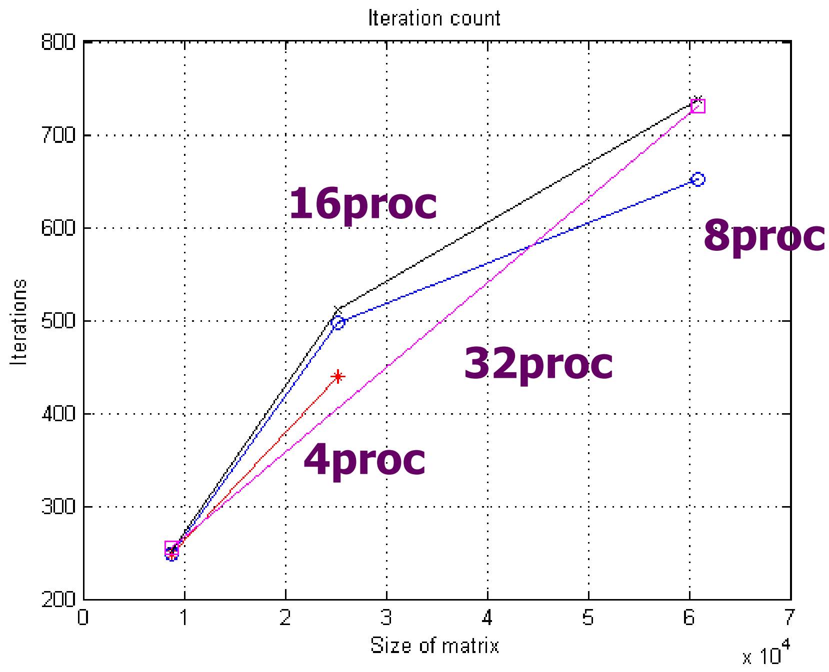
Search for 3 eigenvalues near 500th from bottom.





Convergence behavior





Closure and Outlook

- We have presented a scalable method based on JDQZ and Modified Multigrid to iteratively compute interior eigenvalues of real symmetric generalized systems. The complex symmetric case still requires analysis.

- Modifications and further analysis for faster computation, and verication of effectiveness for 3D calculations is required.

HiQLab group and Software

- PIs: Prof. Sanjay Govindjee (ETHZ)
  Prof. James Demmel (CS and Math, Berkeley)
  Prof. Roger Howe (Stanford)
- Post doctoral students:
  Dr. David Bindel (Courant Institute, NYU)
  Dr. Emmanuel Quevy(Electrical Eng., Berkeley)
- Graduate students:
  Wei He (Civil Engineering, Berkeley)
  Members of the SUGAR group
- HiQLab: Resonant MEMS Simulator
  - http://www.cims.nyu.edu/~dbindel/hiqlab