Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Computing Interior Eigenvalues of a Generalized Complex-Symmetric Pencil arising from the Modeling of Resonant MEMS Systems

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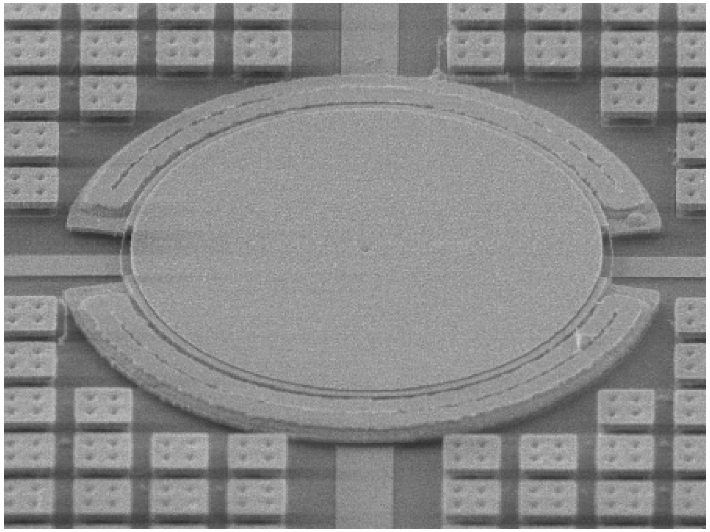
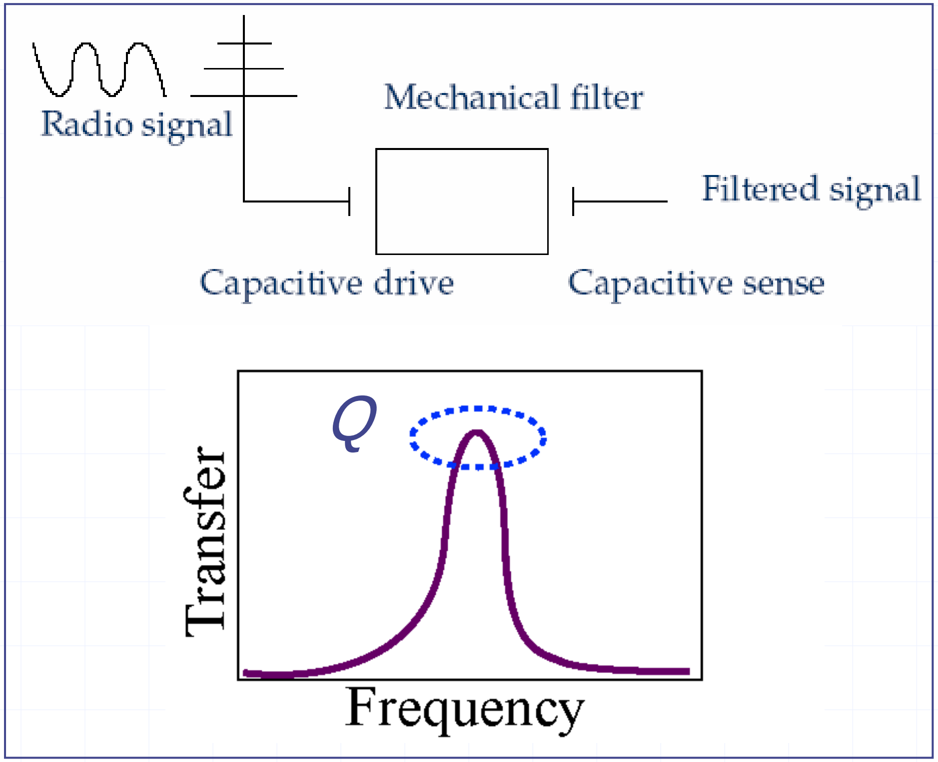
High-frequency MEMS resonators (MHz-GHz)

- Applications as small size, low energy consuming frequency references, filters, and sensors



SEM of 41.5 μm radius poly-SiGe disk resonator



Resonator simulation

- Design requires knowledge of
  - Frequency ← Existing Software and Methods
  - **Quality factor (*Q*)** ← ?

$$Q = \frac{\text{Maximum Stored Energy}}{\text{Energy Loss per radian}} \approx \frac{1}{\text{Damping}}$$

⇩

Tools and methods for evaluating damping in resonant MEMS

Mathematical problem

- Equation of motion discretized with FEM under harmonic assumption

Quadratic eigenvalue problem.

$$\left(-\omega^2 \mathbf{M} + \mathbf{K}\right) \mathbf{x} = 0$$

Complex eigenfrequency

$$\omega = \omega_R + i\omega_I$$

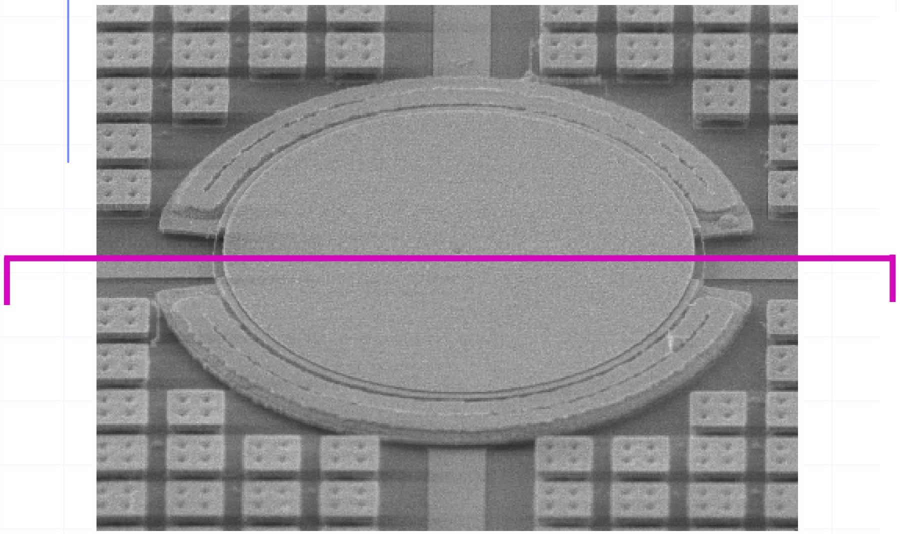
$$Q = \frac{|\omega|}{2\omega_I}$$

Overview

- Characteristics of the eigenvalue problem
  - Complex symmetry from Perfectly Matched Layers
- Solution method
  - Projection methods
  - Jacobi-Davidson QZ method (JDQZ)
  - Geometric multigrid preconditioned GMRES for the solution of the correction equation
- Numerical example of a disk resonator
- Conclusions

Damping mechanism: Anchor loss

- Mechanism: Energy loss from radiating waves escaping into the substrate.



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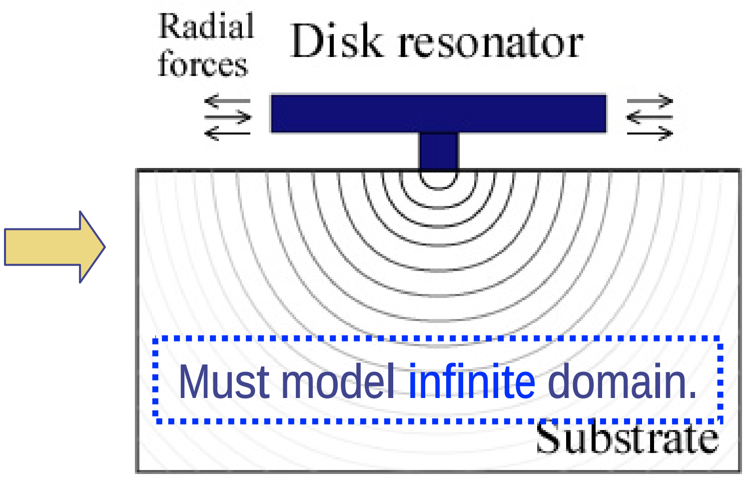
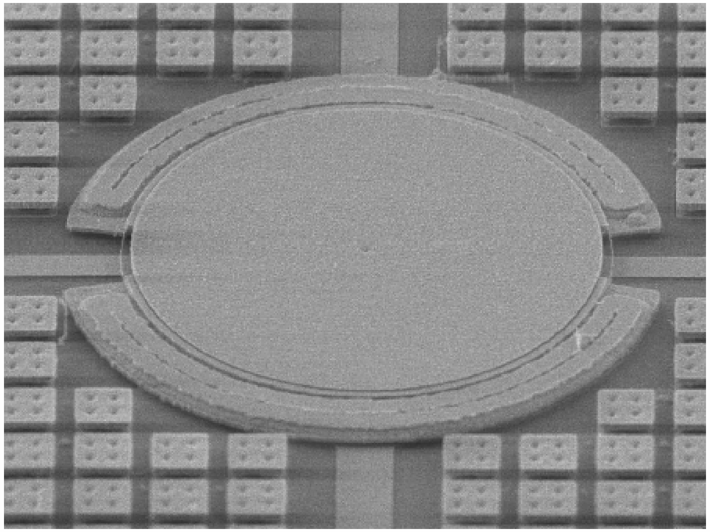


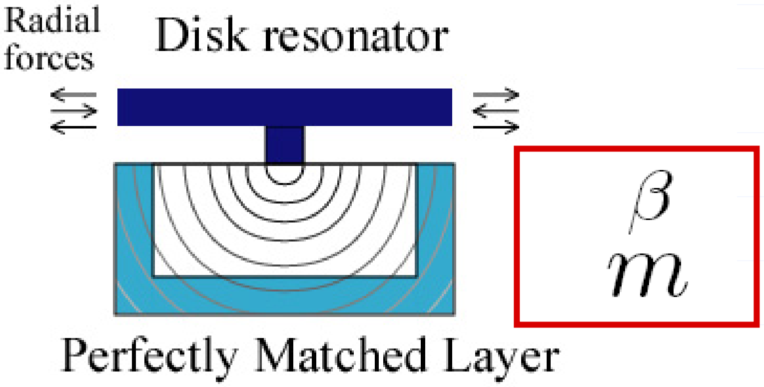
Section of disk resonator

Perfectly Matched Layers (PML)

- Mechanism: Energy loss from radiating waves escaping into the substrate.



SEM of 41.5 μm radius poly-SiGe disk resonator



Outgoing waves are absorbed with zero impedance mismatch at PML boundaries.

Difficulties

- PML implementation:

Generalized eigenvalue problem

$$\mathbf{Kx} = \omega^2 \mathbf{Mx}$$

1. Sparse matrices are large (millions and larger) for accurate results

   Requires large scale sparse-eigenvalue methods.

2. Matrices are complex valued symmetric, Non-Hermitian

   Complex-valued eigenvalues,

3. Desired modes are not at the exterior of the spectrum but interior

   Adds difficulty in the linear solves. Little knowledge about optimal iterative linear system solution methods. (for complex symmetric systems).

Proposed solution

1. Sparse matrices are large (millions and larger) for accurate results

Projection methods.

2. Matrices are complex valued, Non-Hermitian
3. Desired modes are not at the exterior of the spectrum but interior

Elastodynamic equation (SPD pencil without PML) → Geometric multigrid preconditioned GMRES linear solver

Iterative solvers become expensive for higher accuracy → Jacobi-Davidson QZ eigenvalue method

Projection methods

- Find eigen pair $(\omega^2, \mathbf{x} \in \mathcal{V})$ such that

$$\left(\mathbf{K} - \omega^2 \mathbf{M}\right) \mathbf{x} \perp \mathcal{W}$$

- Galerkin projection(Ritz values,vectors) $\mathcal{W} = \mathcal{V}$
- Petrov-Galerkin projection $\mathcal{W} \neq \mathcal{V}$

Select harmonic-Ritz projection for better approximation of interior eigenvalues.

$$\mathcal{W} = \left(\mathbf{K} - \omega_0^2 \mathbf{M}\right) \mathcal{V}$$

Jacobi-Davidson QZ (JDQZ)

- Harmonic JD method for $(\mathbf{K}, \mathbf{M})$ target ω_0^2

Given subspace $\mathcal{V}$, $\mathcal{W} = \left(\mathbf{K} - \omega_0^2 \mathbf{M}\right) \mathcal{V}$

1. Find approximate eigenpair $(\omega^2, \mathbf{u})$

$$\mathbf{u} \in \mathcal{V} \quad \left(\mathbf{K} - \omega^2 \mathbf{M}\right) \mathbf{u} \perp \mathcal{W}$$

2. Solve correction equation for $\mathbf{t}$

3. Expand subspace

$$\mathcal{V}_{\text{new}} = \mathcal{V} \oplus \text{span}\{\mathbf{t}\}$$
$$\mathcal{W}_{\text{new}} = \mathcal{W} \oplus \text{span}\left\{\left(\mathbf{K} - \omega_0^2 \mathbf{M}\right) \mathbf{t}\right\}$$

(Fokkema, Sleijpen, and van der Horst, 1998)

Correction equation

- Find $\mathbf{t} \perp \mathbf{u}$ such that $(\mathbf{u}+\mathbf{t})$ is a better approximation to the desired eigenvector

$$\left(\mathbf{I}-\mathbf{p}\mathbf{p}^{*}\right)\left(\mathbf{K}-\omega^{2}\mathbf{M}\right)\left(\mathbf{I}-\mathbf{u}\mathbf{u}^{*}\right)\mathbf{t}=-\mathbf{r}$$

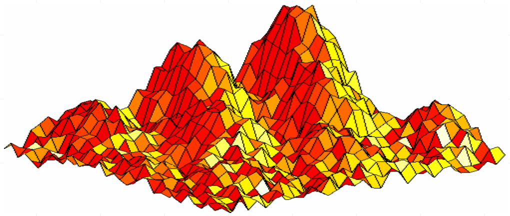
$$\begin{aligned} \mathbf{r} &= \left(\mathbf{K}-\omega^{2}\mathbf{M}\right)\mathbf{u} \\ \mathbf{p} &= \left(\mathbf{K}-\omega_{0}^{2}\mathbf{M}\right)\mathbf{u} \in \mathcal{W} \end{aligned}$$

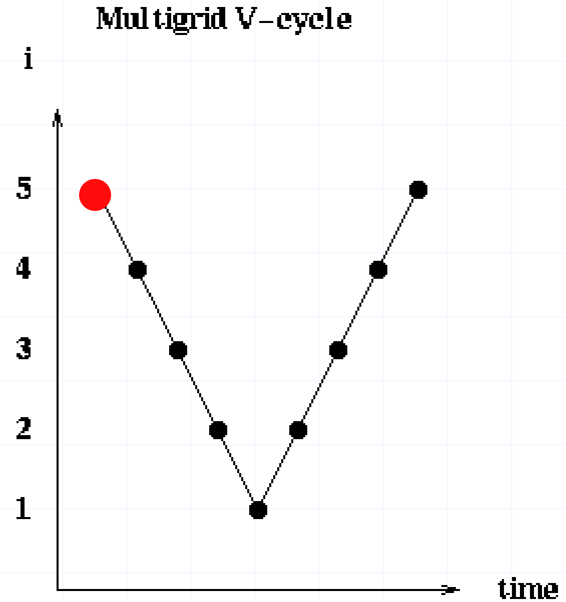
Must solve linear systems of the form

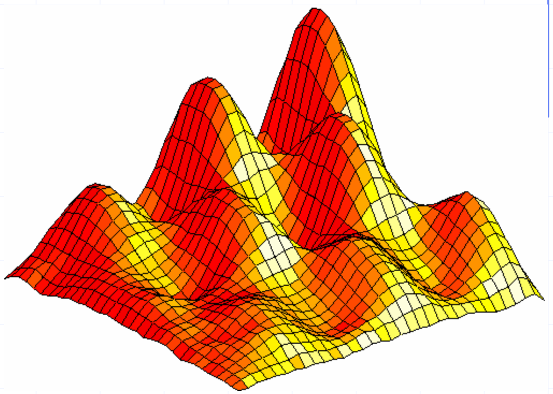
$$\left(\mathbf{K}-\omega^{2}\mathbf{M}\right)\mathbf{x}=\mathbf{b}$$

⇨ Geometric multigrid preconditioned GMRES to non-Hermitian complex-valued linear system

Geometric multigrid

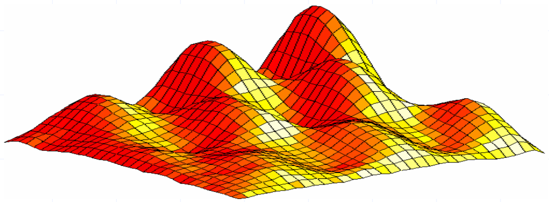






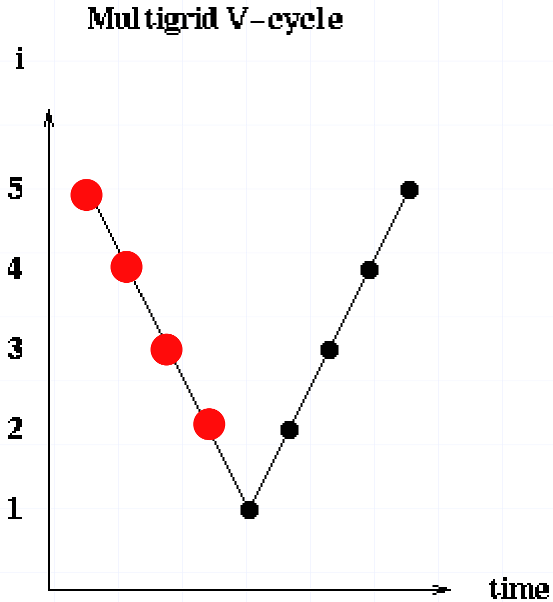
(Demmel, Parallel Computing Notes)

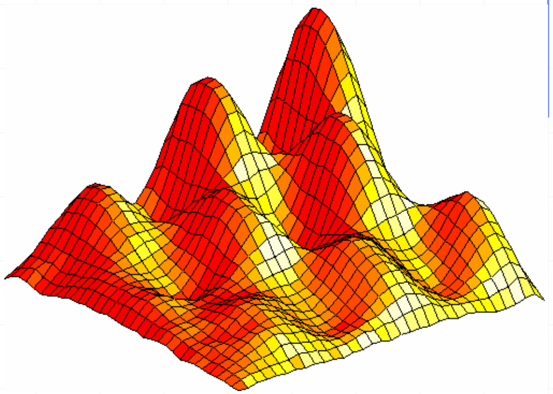
Geometric multigrid



Smoother

Remove error with large eigenvalue





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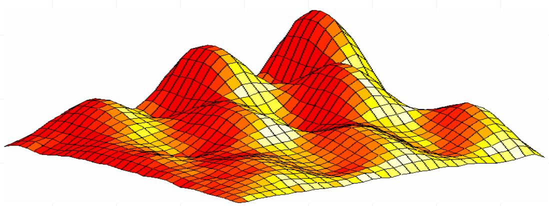

Smoother

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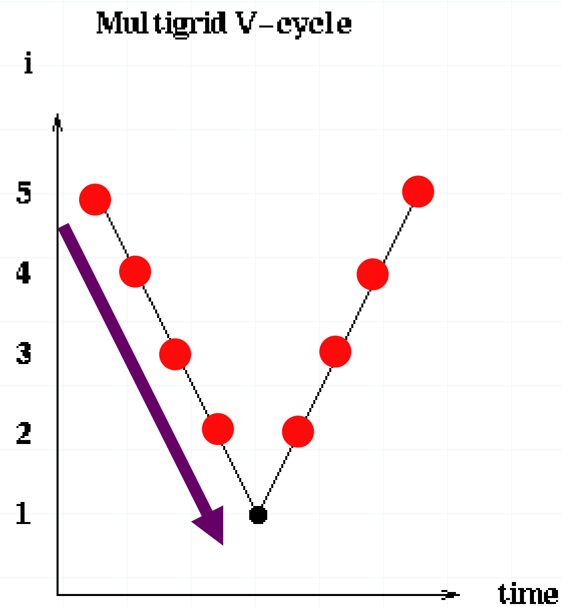

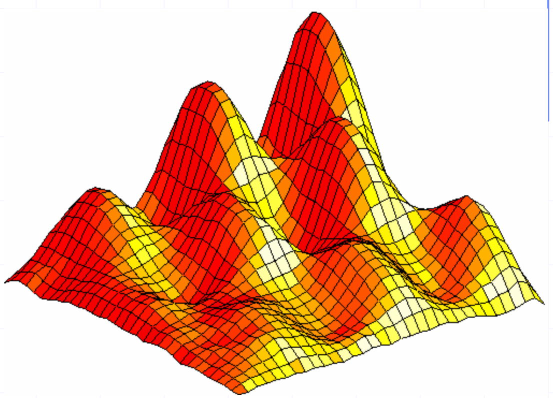

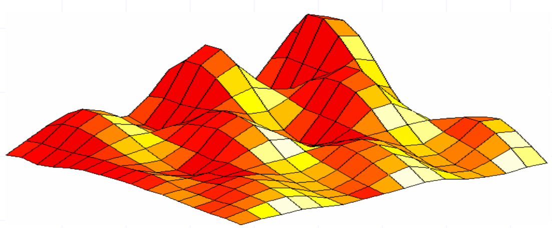

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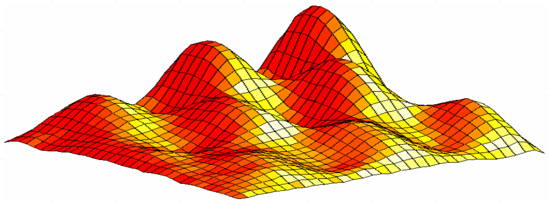
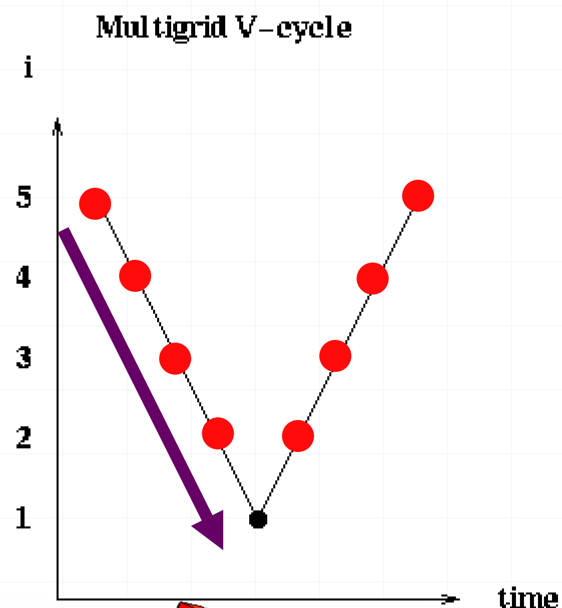
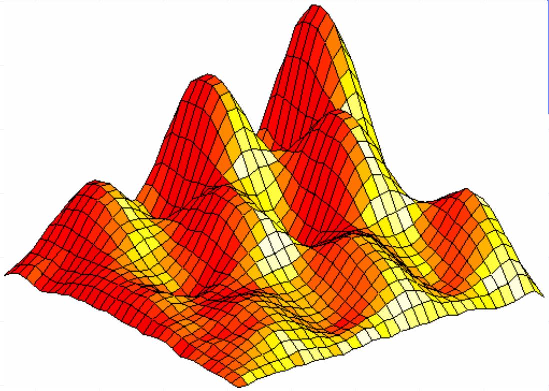
Geometric multigrid

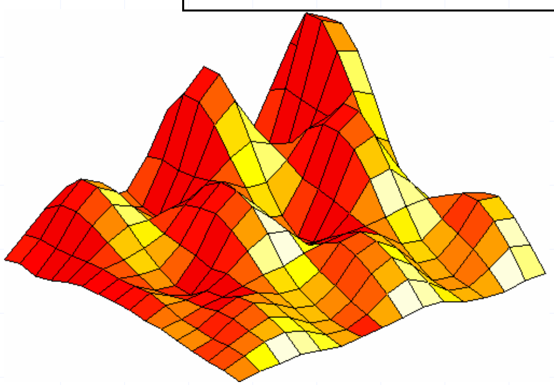


Smoother

Remove error with large eigenvalue







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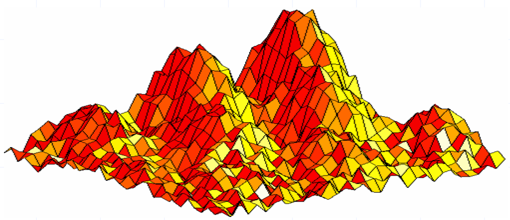
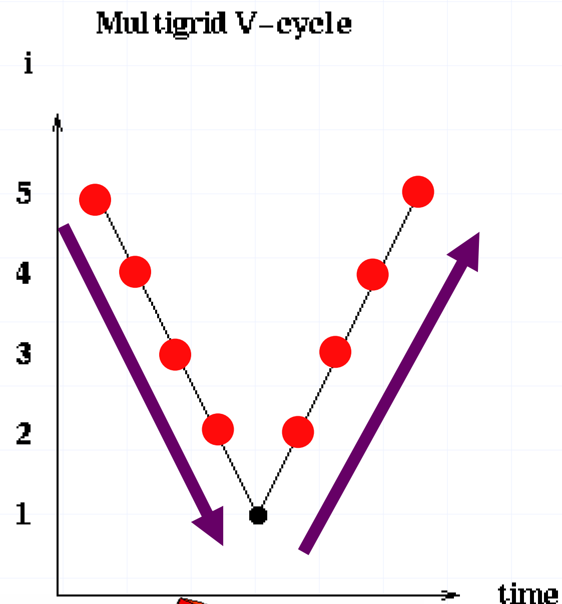
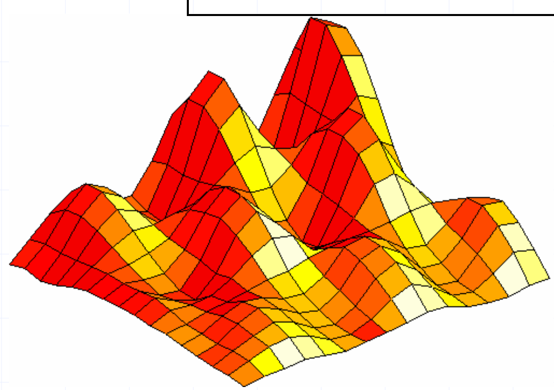
Geometric multigrid

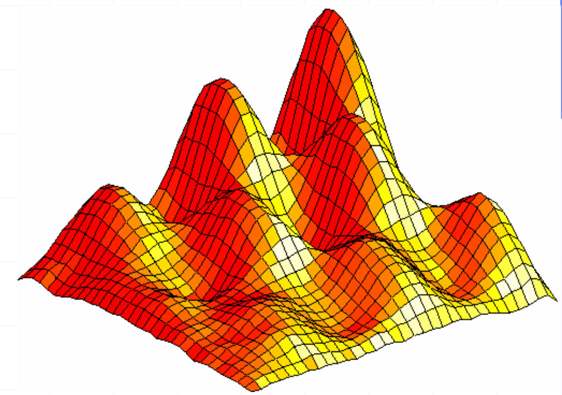


Smoother

Remove error with large eigenvalue







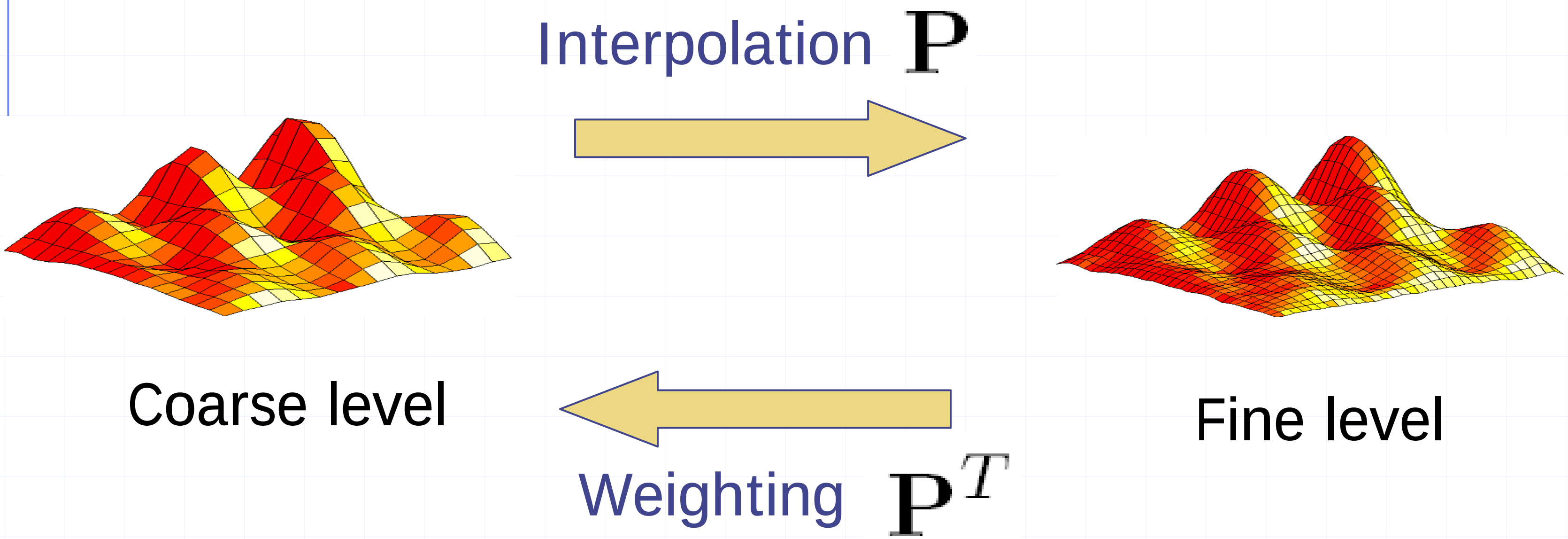
Coarse grid correction (Prolongation operator)

Remove error with small eigenvalue

(Demmel, Parallel Computing Notes)

Prolongation operators

- Maps solutions from one level to another with geometric information



Smoother for $\mathbf{A}\mathbf{x} = \mathbf{b}$

- No heuristics for choosing appropriate smoother for complex-valued non-Hermitian case

⇒ Gauss-Seidel (Restriction in PML parameter selection.)

PML parameters

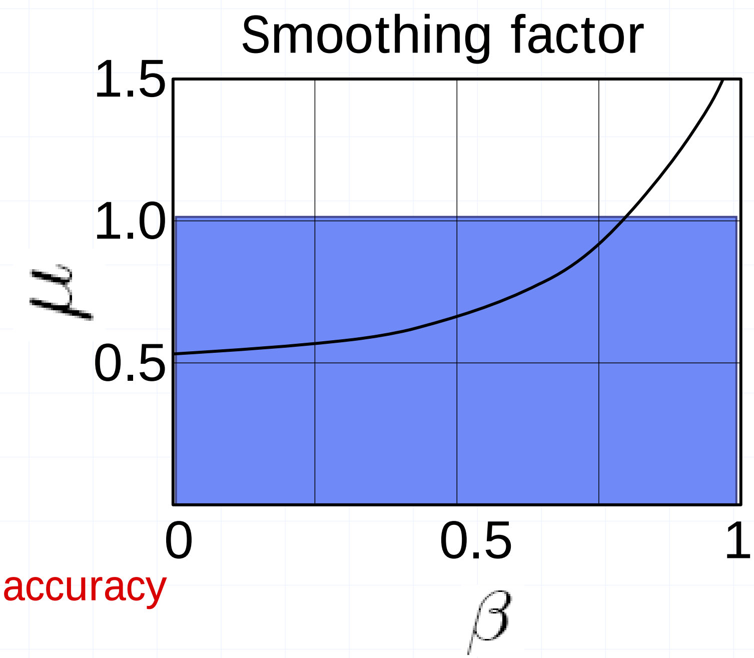
β	: Absorption strength
m	: Wave lengths in PML

For convergent smoother, $0 \leq \mu < 1$

⇒ β small Restriction for solvability

For small wave reflection

⇒ m approx. 1-2 Restriction for accuracy



Two grid convergence factor

- Error reduction for multigrid with 2-levels

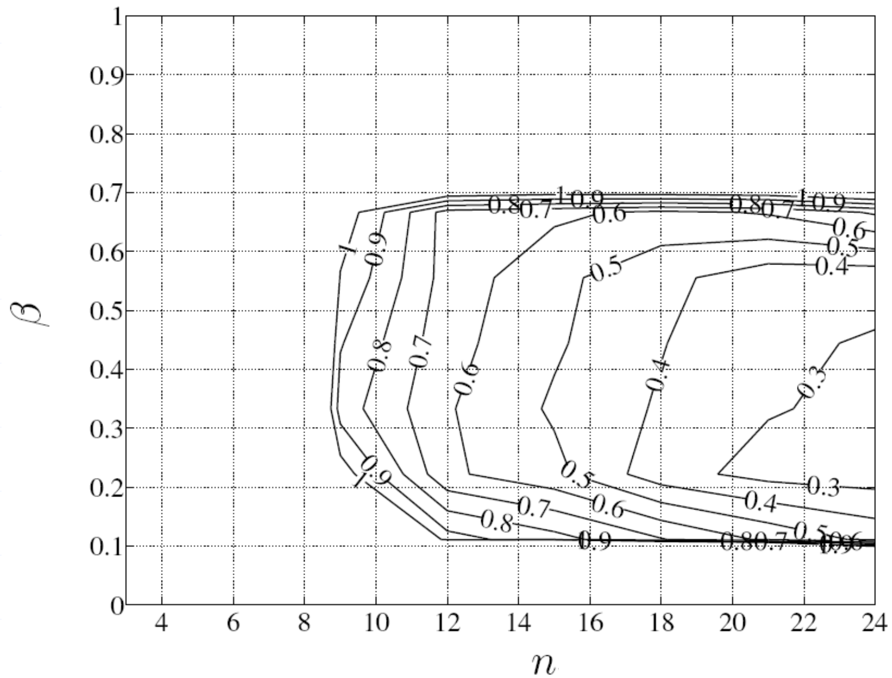
Fine level: n per wave

Coarse level: n /2 per wave

β	: Absorption strength
n	: Nodes per wave on fine grid(discretization)

β at most 1
At least 6 nodes per wave on coarse grid

Gauss-Seidel



Restriction for multigrid convergence

JDQZ + GMG preconditioned GMRES

- The performance of JDQZ depends strongly on initial eigenvalue/eigenvector approximation

⇨ Use coarse grid solutions as initial starting points.

Interpolation with geometric multigrid prolongation operator

Coarser level converged solution

$\mathbf{P}$

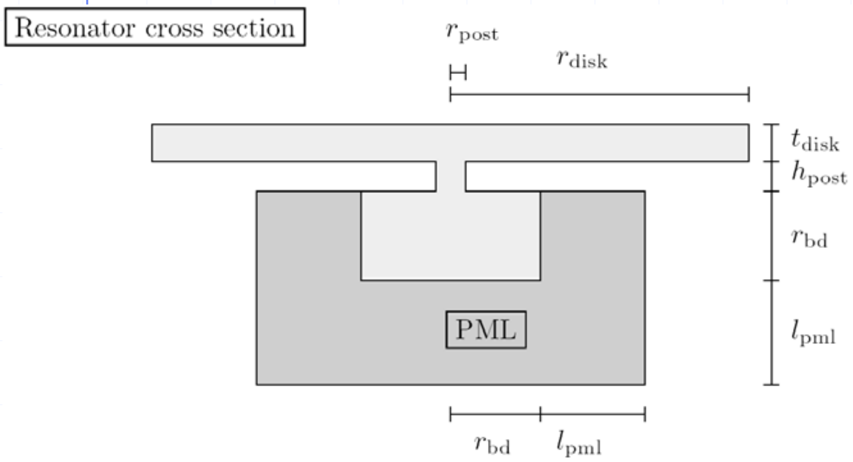
Desired level initial guess

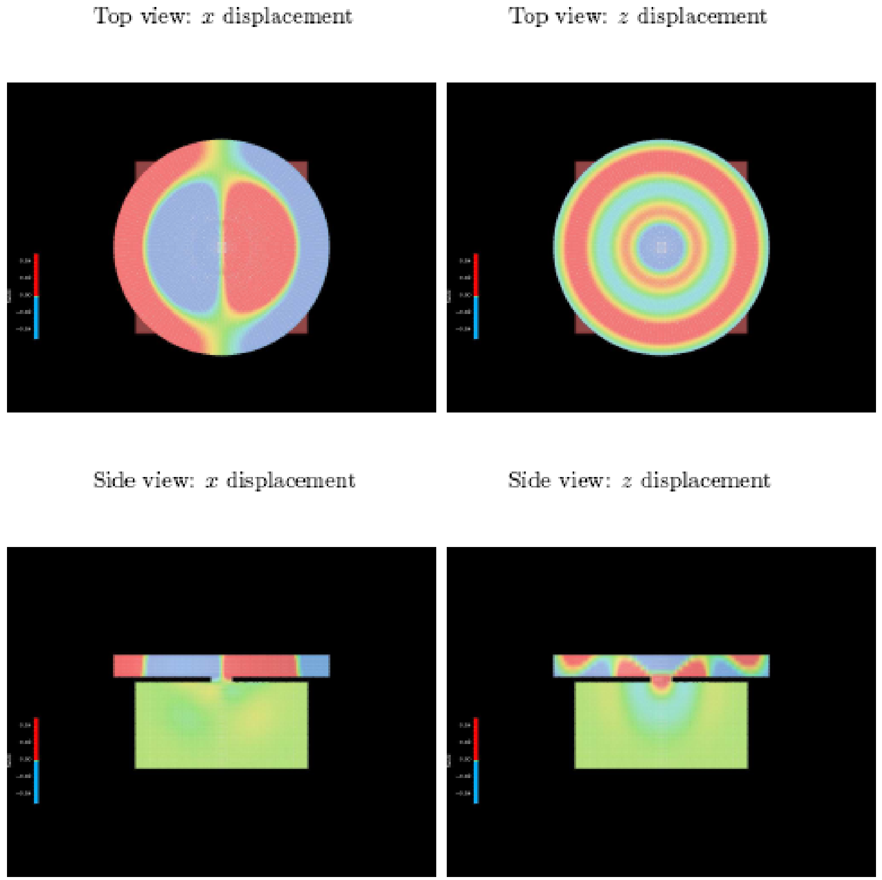
$$(\omega_{\text{coarse}}, \mathbf{u}_{\text{coarse}}) \longrightarrow (\omega_{\text{coarse}}, \mathbf{P}\mathbf{u}_{\text{coarse}})$$

JDQZ converges in 1-2 iterations (correction equation solves) to tolerance 1e-10 with good initial guess. Scalability of iterative method is crucial.

Numerical example: Disk resonator

- 2nd radial contour mode at 715 [MHz]





Scalability of iterative method (PETSc implementation)

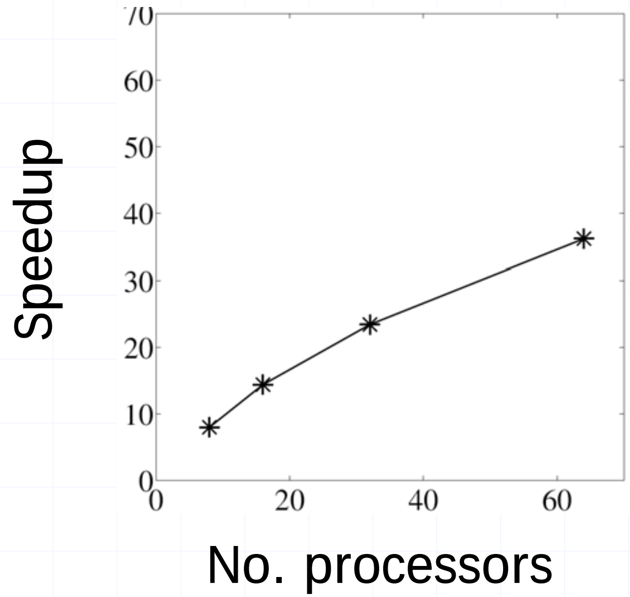
- Scaling with respect to size

Level	Number of DOFS
1	49938
2	197574
3	977115
4	6140520

GMRES iters. for preconditioned residual of 1×10^{-10} for $\beta = 1.0$

Levels	1-2	1-3	1-4
715.7[MHz] (2nd mode)	49(1.2e-09)	51(1.6e-09)	41(3.3e-09)

- Speed-up with respect to number of processors



Conclusions

- An algorithm for solving complex-symmetric eigenvalue problems arising from mechanical systems with Perfectly Matched Layers (PML)

- Jacobi-Davidson QZ + geometric multigrid
  - scalable, fast,
    - converging in 1-2 iterations with coarse grid initial values
  - restrictions on PML parameters
    - converging smoother: β small
    - converging multigrid : at least 6 nodes per wave on coarsest grid
    - numerical accuracy : approx. 1-2 wave lengths in PML